The reality of the quantum state from Kochen-Specker contextuality

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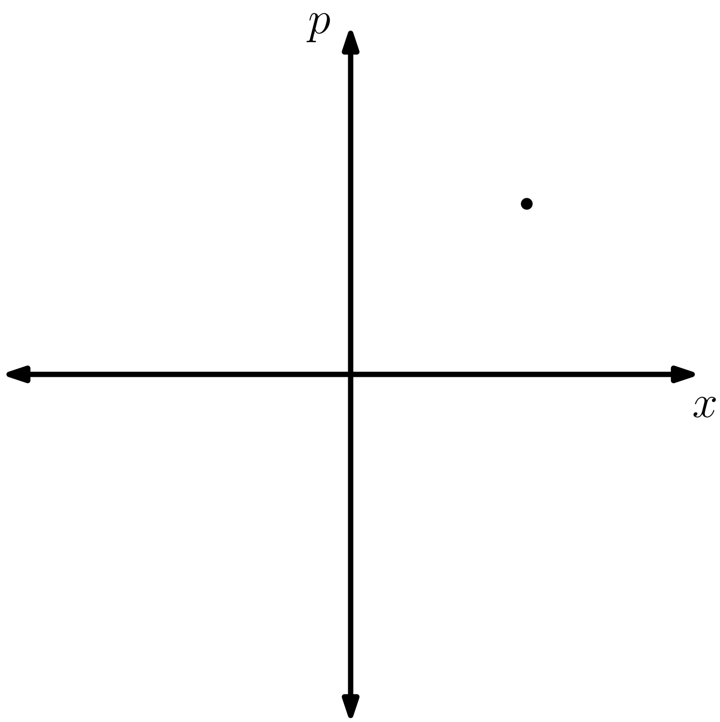
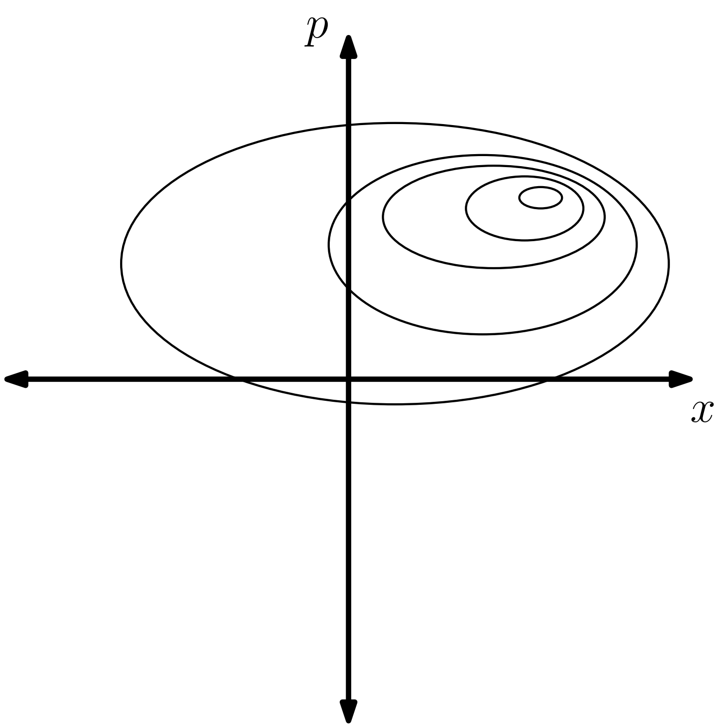
Introduction

- *Ontic state*: a state of reality.
  - ψ-*ontic*: the quantum state is ontic.
- *Epistemic state*: a state of knowledge or information.
  - ψ-*epistemic*: the quantum state is epistemic.

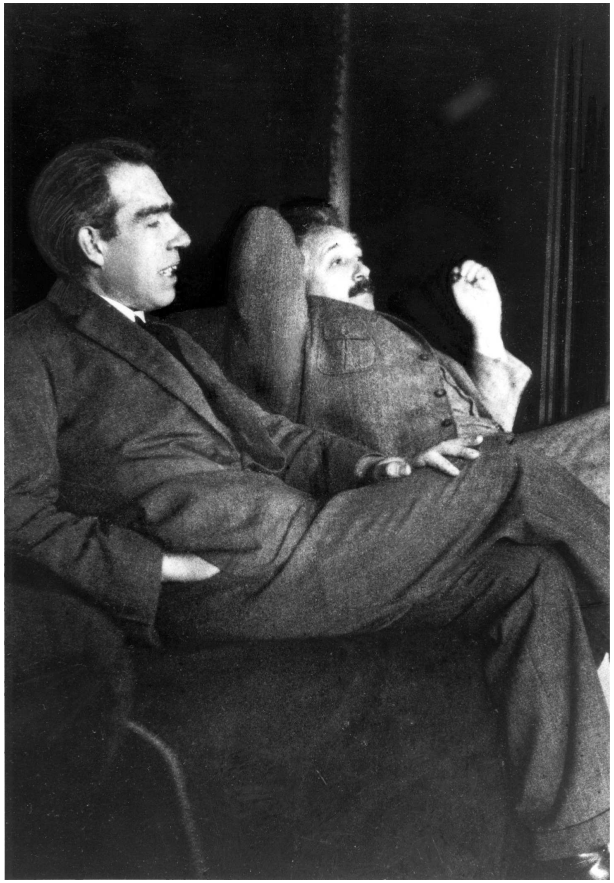
Ontic state



Epistemic state



Bohr and Einstein: ψ-epistemicists



Source: http://en.wikipedia.org/

> There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature. — Niels Bohr[a]

> [t]he ψ-function is to be understood as the description not of a single system but of an ensemble of systems. — Albert Einstein[b]

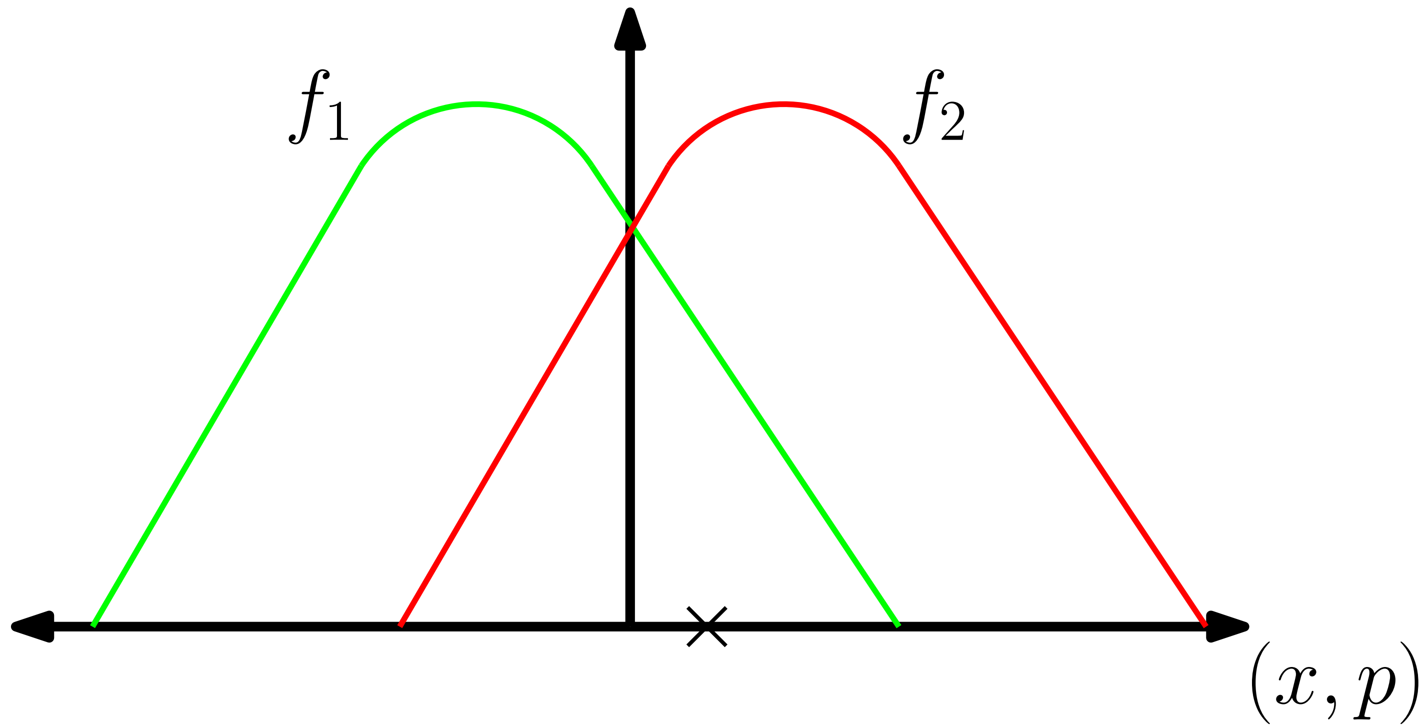
[a] Quoted in A. Petersen, "The philosophy of Niels Bohr", *Bulletin of the Atomic Scientists* Vol. 19, No. 7 (1963)

[b] P. A. Schilpp, ed., *Albert Einstein: Philosopher Scientist* (Open Court, 1949)

Interpretations of quantum theory

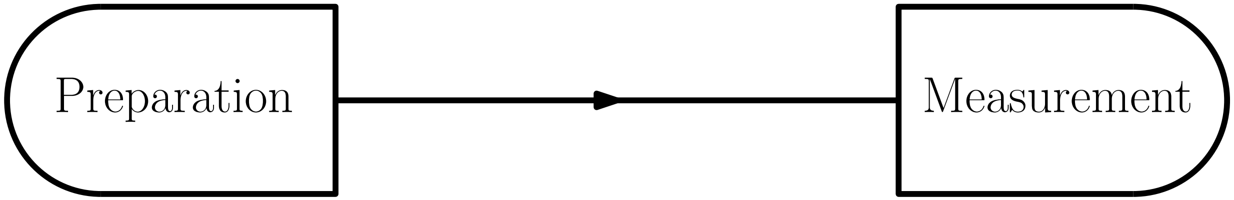
	ψ-epistemic	ψ-ontic
Anti-realist	Copenhagen neo-Copenhagen (e.g. QBism, Peres, Zeilinger, Healey)	
Realist	Einstein Ballentine? Spekkens ?	Dirac-von Neumann Many worlds Bohmian mechanics Spontaneous collapse Modal interpretations

Epistemic states overlap



Ontological Models

Prepare-and-measure experiments: Quantum description

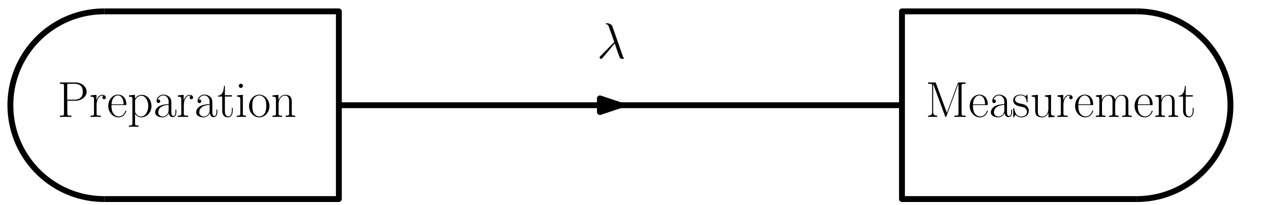


$|\psi\rangle \in \mathbb{C}^d$

$M = \{|a\rangle, |b\rangle, \ldots\}$

$$\text{Prob}(a|\psi, M) = |\langle a|\psi\rangle|^2$$

Prepare-and-measure experiments: Ontological description



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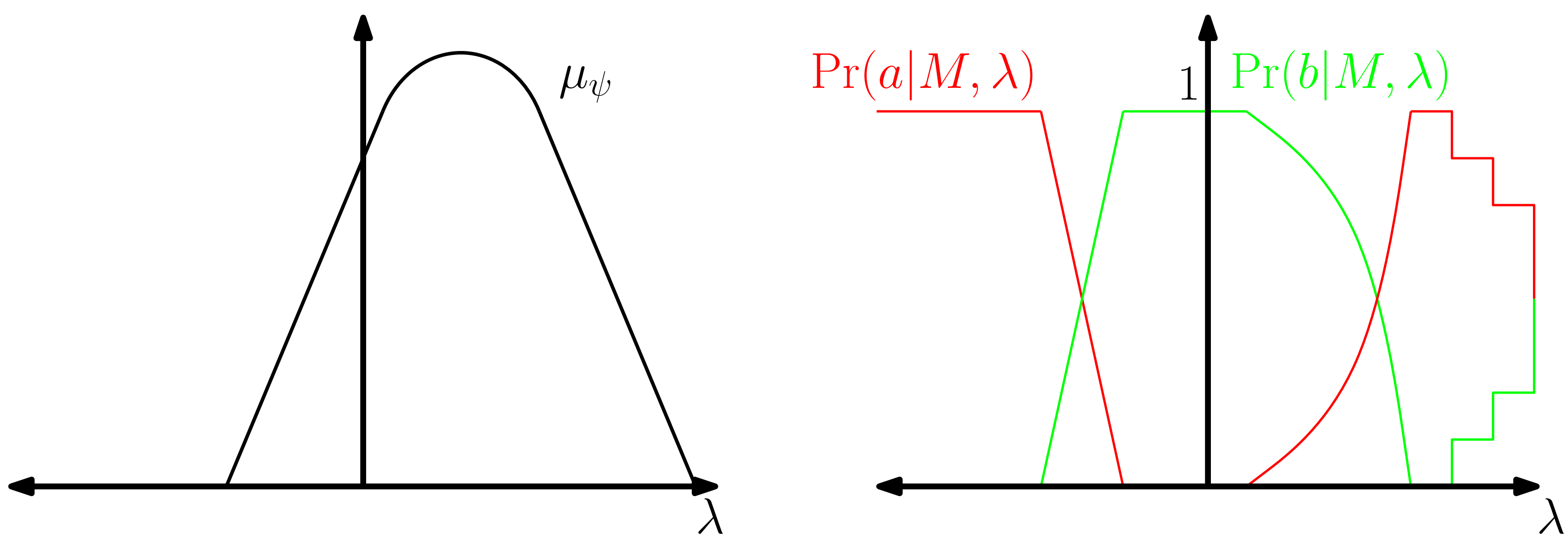
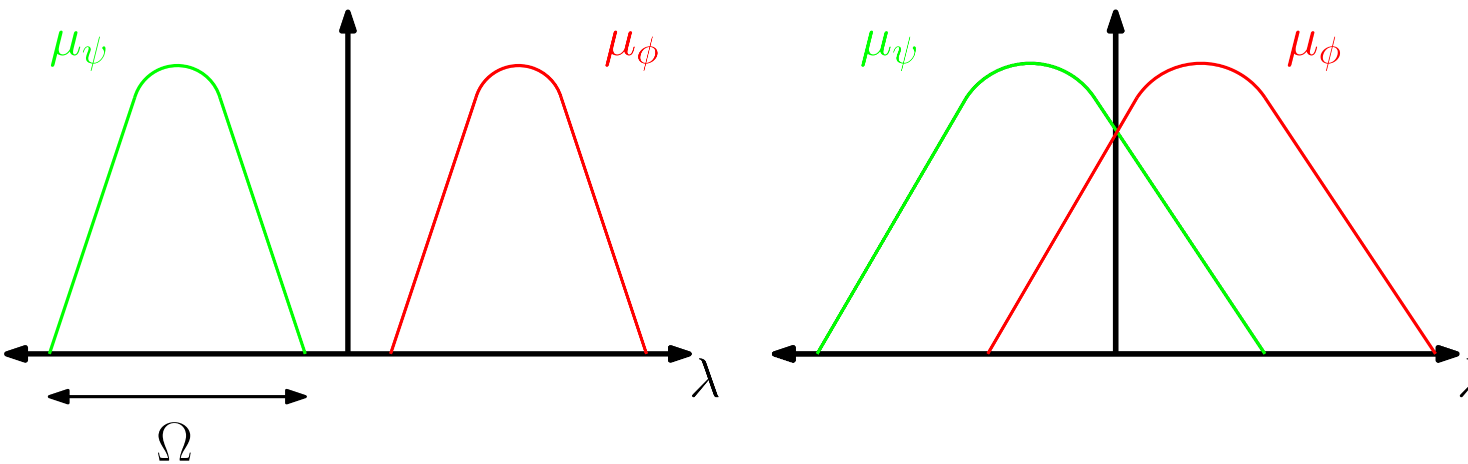


$$\text{Prob}(a|\psi, M) = \int \text{Pr}(a|M,\lambda) d\mu_\psi$$

- $|\psi\rangle$ and $|\phi\rangle$ are *ontologically distinct* in an ontological model if there exists $\Omega \in \Sigma$ s.t.

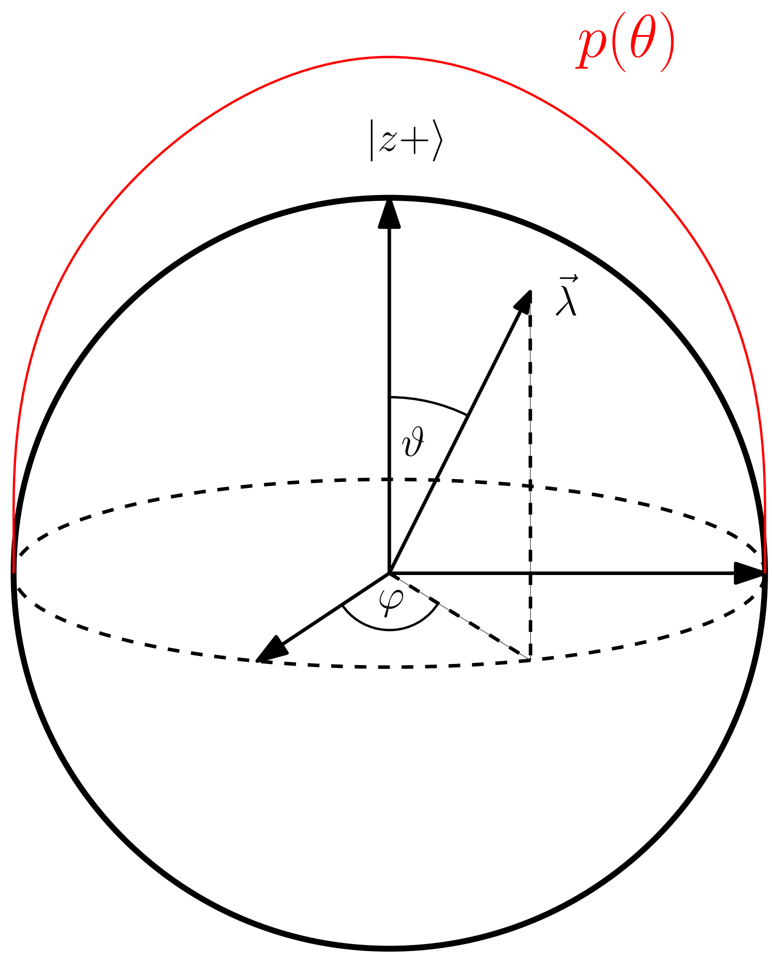
$$\mu_\psi(\Omega) = 1 \qquad \mu_\phi(\Omega) = 0.$$



- An ontological model is *ψ-ontic* if every pair of states is ontologically distinct. Otherwise it is *ψ-epistemic*.

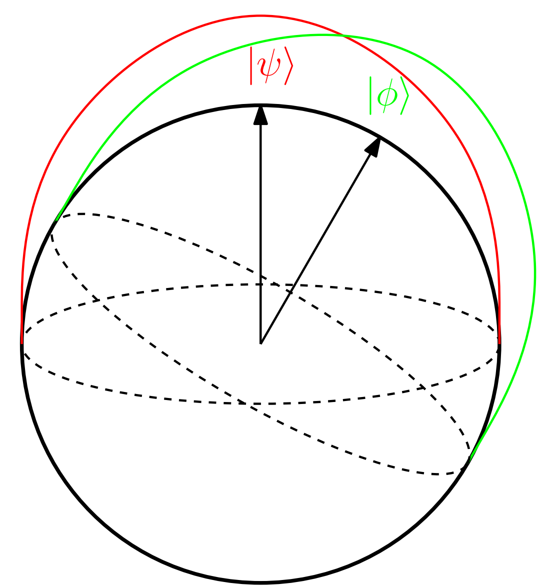
ψ-ontology theorems

- The Colbeck-Renner theorem: R. Colbeck and R. Renner, arXiv:1312.7353 (2013).
- Hardy's theorem: L. Hardy, *Int. J. Mod. Phys. B*, 27:1345012 (2013) arXiv:1205.1439
- The Pusey-Barrett-Rudolph theorem: M. Pusey et. al., *Nature Physics*, 8:475–478 (2012) arXiv:1111.3328



$$\mu_{z+}(\Omega) = \int_{\Omega} p(\vartheta) \sin\vartheta d\vartheta d\varphi$$

$$p(\vartheta) = \begin{cases} \frac{1}{\pi}\cos\vartheta, & 0 \leq \vartheta \leq \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \vartheta \leq \pi \end{cases}$$



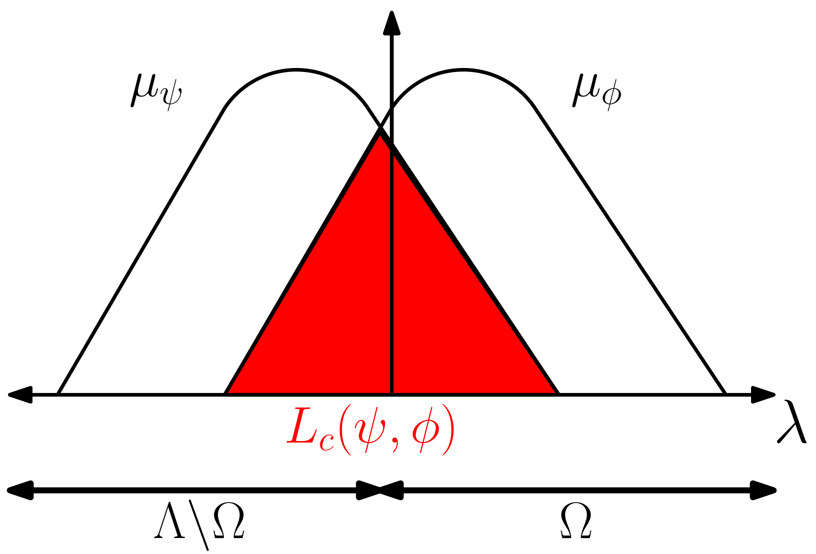
S. Kochen and E. Specker, *J. Math. Mech.*, 17:59–87 (1967)

- Lewis et. al. provided a ψ-epistemic model for all finite d.
  - P. G. Lewis et. al., *Phys. Rev. Lett.* 109:150404 (2012) arXiv:1201.6554
- Aaronson et. al. provided a similar model in which every pair of nonorthogonal states is ontologically indistinct.
  - S. Aaronson et. al., *Phys. Rev. A* 88:032111 (2013) arXiv:1303.2834
- These models have the feature that, for a fixed inner product, the amount of overlap decreases with d.

Overlap bounds

- *Classical overlap*:

$$L_c(\psi, \phi) := \inf_{\Omega \in \Sigma} \left[\mu_\psi(\Omega) + \mu_\phi(\Lambda \backslash \Omega)\right]$$



- Optimal success probability of distinguishing $|\psi\rangle$ and $|\phi\rangle$ if you know λ:

$$p_c(\psi, \phi) = \frac{1}{2}\left(2 - L_c(\psi, \phi)\right)$$

Quantum overlap

- *Classical overlap*:

$$L_c(\psi, \phi) := \inf_{\Omega \in \Sigma} \left[\mu_\psi(\Omega) + \mu_\phi(\Lambda \backslash \Omega)\right]$$

- *Quantum overlap*:

$$L_q(\psi, \phi) := \inf_{0 \leq E \leq I} \left[\langle\psi| E |\psi\rangle + \langle\phi| (I - E) |\phi\rangle\right]$$
$$= 1 - \sqrt{1 - |\langle\phi|\psi\rangle|^2}$$

- Optimal success probability of distinguishing $|\psi\rangle$ and $|\phi\rangle$ based on a quantum measurement:

$$p_q(\psi, \phi) = \frac{1}{2}\left(2 - L_q(\psi, \phi)\right)$$

- Given a set V of states, and another state $|\psi\rangle$, we can upper bound the average overlap

$$\langle L_c \rangle = \frac{1}{|V|} \sum_{|a\rangle \in V} L_c(\psi, a).$$

- Most works use this to bound the ratio:

$$k = \frac{\langle L_c \rangle}{\langle L_q \rangle}.$$

- Better to use the difference:
  - *Overlap deficit*: $\Delta L = \langle L_q \rangle - \langle L_c \rangle$

Previous results

	Dimension	$\|V\|$	$\langle L_c \rangle$	$\langle L_q \rangle$
Barrett et. al.[1]	Prime power $d \geq 4$	d^2	$1/d^2$	$1 - \sqrt{1 - 1/d}$
Leifer[2] (Branciard's version)[3]	$d \geq 3$	2^{d-1}	$1/2^{d-1}$	$1 - \sqrt{1 - 1/d}$
Branciard[3]	$d \geq 4$	$n \geq 2$	$1/n$	$1 - \sqrt{1 - \frac{1}{4} n^{-1/(d-2)}}$

[1] J. Barrrett et. al., Phys. Rev. Lett. 112, 250403 (2014)
[2] ML, Phys. Rev. Lett. 112, 160404 (2014)
[3] C. Branciard, Phys. Rev. Lett. 113, 020409 (2014)

Optimizing for distinguishability deficit

	Optimal dimension	Optimal $\lvert V \rvert$	ΔL
Barrett et. al.	4	16	0.0715
Leifer (Branciard's version)	5	16	0.385
Branciard	4	$n \to \infty$	0.134

- Ringbauer et. al.[4] experiment (based on Branciard's construction) obtained:

$$k \leq 0.690 \pm 0.001$$

$$\Delta L \geq 0.047 \pm 0.010$$

- My analysis suggests larger ΔL should be obtainable from my construction (with Branciard's analysis).

[4] M. Ringbauer et. al. Nature Physics 11, 249–254 (2015).

Overlap bounds from contextuality

Kochen-Specker noncontextuality

- Let $\mathcal{M}$ be a set of orthonormal bases in $\mathbb{C}^d$.
- An ontological model for $\mathcal{M}$ is *Kochen Specker noncontextual* if it is
  - *Outcome deterministic*: $\text{Pr}(a|M,\lambda) \in \{0,1\}$
  - *Measurement noncontextual*: If there exist $M, N \in \mathcal{M}$ and $|a\rangle$ such that $|a\rangle \in M$ and $|a\rangle \in N$ then

$$\text{Pr}(a|M,\cdot) = \text{Pr}(a|N,\cdot).$$

- Define:

$$\Gamma_a^M = \{\lambda \in \Lambda | \text{Pr}(a|M,\lambda) = 1\} \qquad \Gamma_a = \bigcap_{\{M \in \mathcal{M} | |a\rangle \in M\}} \Gamma_a^M$$

Theorem: There exists a KS noncontextual model for $\mathcal{M}$ iff there exists a model where, for all $|\psi\rangle$, $M \in \mathcal{M}$, $|a\rangle \in M$,

$$\int_\Lambda \text{Pr}(a|M,\lambda) d\mu_\psi(\lambda) = \mu_\psi(\Gamma_a).$$

- For a (finite) set V of states, a noncontextuality inequality is a bound of the form

$$\sum_{|a\rangle \in V} \mu_\psi(\Gamma_a) \leq \gamma.$$

- *Theorem*:

$$L_c(\psi, a) \leq \mu_\psi(\Gamma_a)$$

- Any noncontextuality inequality can be used to derive an overlap bound.

- Other methods for deriving overlap bounds turn out to be noncontextuality inequalities in disguise.

Conclusions

- Summary:
  - Several bounds exist showing $k \to 0$. Harder to get $\Delta L \approx 1$. Best current bound is $\Delta L \approx 0.385$.
  - Any noncontextuality inequality is an overlap bound.
  - Methods developed to bound overlaps yield new contextuality inequalities, sometimes with much tighter bounds.
- Open questions:
  - Error analysis for arbitrary noncontextuality-based overlap bounds.
  - What is the best possible bound on ΔL?
  - Applications in quantum information.

- Become neo-Copenhagen.
- Adopt a more exotic ontology:
  - Nonstandard logics and probability theories.
  - Ironic many-worlds.
  - Retrocausality.
  - Relationalism.

What now for ψ-epistemicists?

- Become neo-Copenhagen.
- Adopt a more exotic ontology:
  - Nonstandard logics and probability theories.
  - Ironic many-worlds.
  - Retrocausality.
  - Relationalism.
- Explanatory conservatism: If there is a natural explanation for a quantum phenomenon then we should adopt an interpretation that incorporates it.
  - Suggests exploring exotic ontologies.

- Review articles:
  - ML, “Is the quantum state real? An extended review of ψ-ontology theorems”, *Quanta* 3:67–155 (2014), arXiv:1409.1570.
  - D. Jennings and ML, “No Return to Classical Reality”, *Contemp. Phys.* 56 (2015). arXiv:1501.03202.
- Overlap bounds and contextuality:
  - ML and O. Maroney, “Maximally epistemic interpretations of the quantum state and contextuality”, *Phys. Rev. Lett.* 110:120401 (2013) arXiv:1208.5132.
  - ML, “ψ-epistemic models are exponentially bad at explaining the distinguishability of quantum states ”*Phys. Rev. Lett.* 112:160404 (2014) arXiv:1401.7996.
  - ML, “Bounds on the epistemic interpretation of the quantum state from contextuality inequalities” in preparation.

Additional slides

Penrose: ψ-ontologist

It is often asserted that the state-vector is merely a convenient description of ‘our knowledge’ concerning a physical system—or, perhaps, that the state-vector does not really describe a single system but merely provides probability information about an ‘ensemble’ of a large number of similarly prepared systems. Such sentiments strike me as unreasonably timid concerning what quantum mechanics has to tell us about the *actuality* of the physical world. — Sir Roger Penrose[5]

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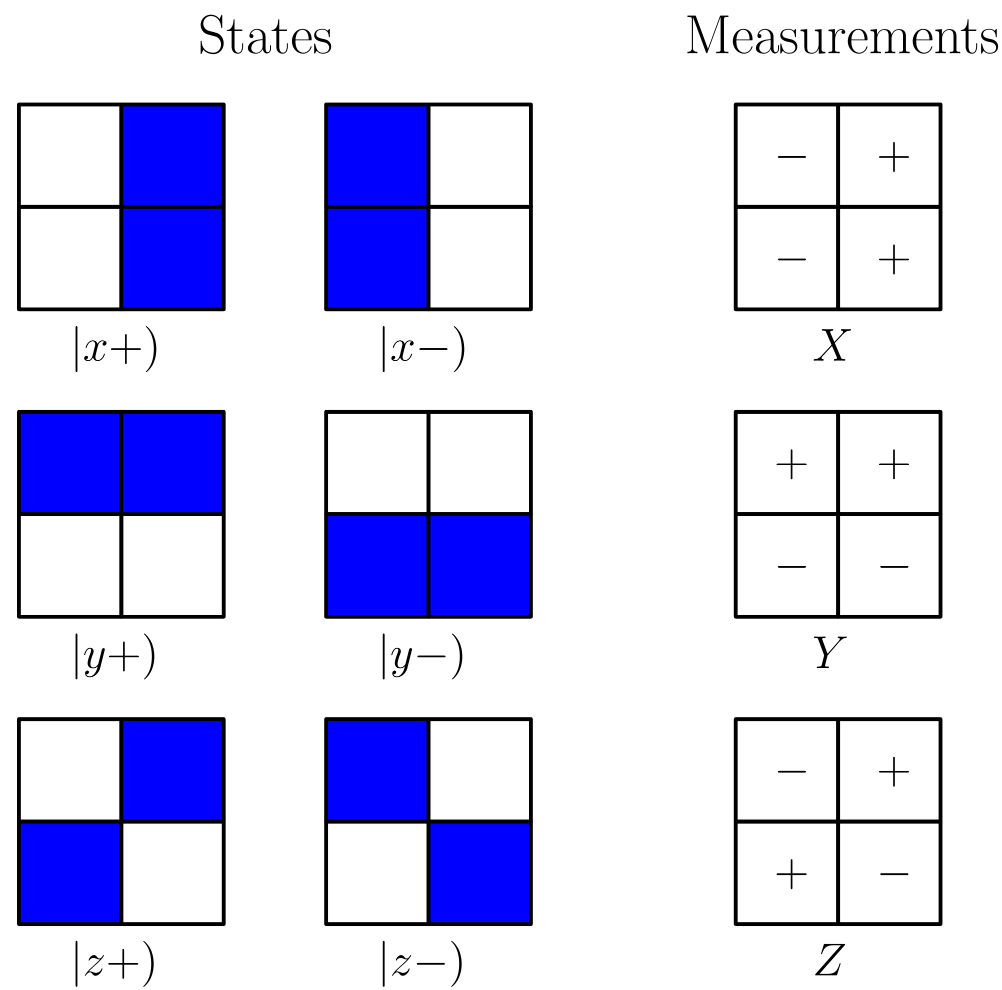
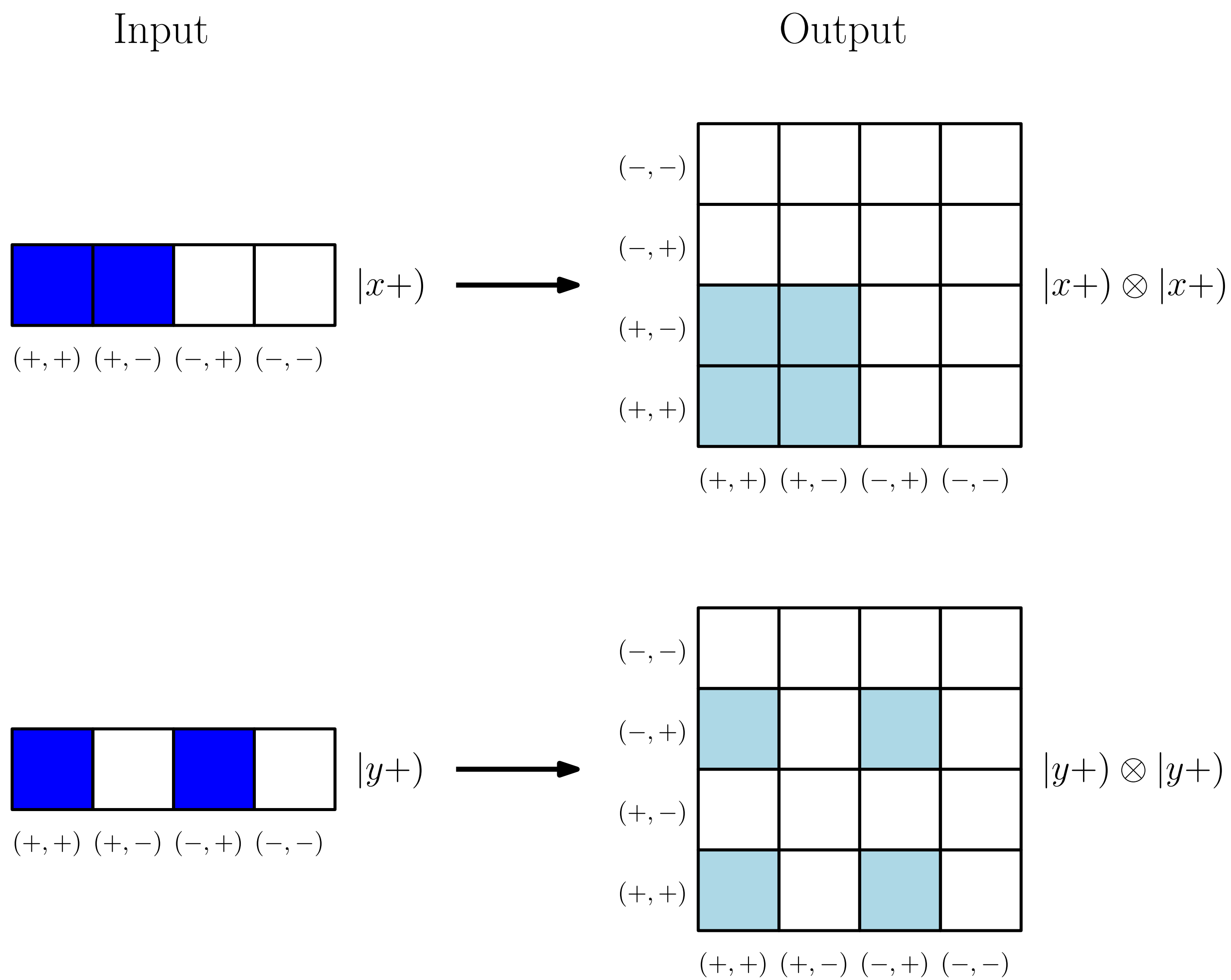
[5] R. Penrose, *The Emperor’s New Mind* pp. 268–269 (Oxford, 1989)

Spekkens' toy theory



R. W. Spekkens, *Phys. Rev. A* 75(3):032110 (2007) arXiv:quant-ph/0401052

No cloning



Other arguments

- Collapse of the wavefunction
- Generalized probability theory
- Excess baggage

See ML Quanta 3:67–155 (2014) for more details.

Arguments for ontic quantum states

- Interference
- Eigenvalue-eigenstate link
- Lack of imagination
- Quantum computing

See ML Quanta 3:67–155 (2014) for more details.

Formal definition

An ontological model for $\mathbb{C}^d$ consists of:

- A measurable space (Λ, Σ).

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- For each orthonormal basis $M = \{|a\rangle, |b\rangle, \ldots\}$, a set of conditional probability functions $\text{Pr}(a|M, \cdot) : \Lambda \to [0,1]$ satisfying

$$\forall M, \lambda, \sum_{|a\rangle \in M} \text{Pr}(a|M, \lambda) = 1.$$

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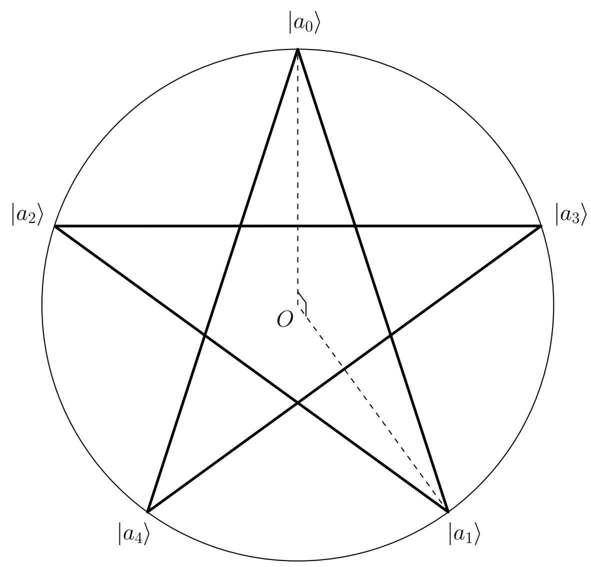
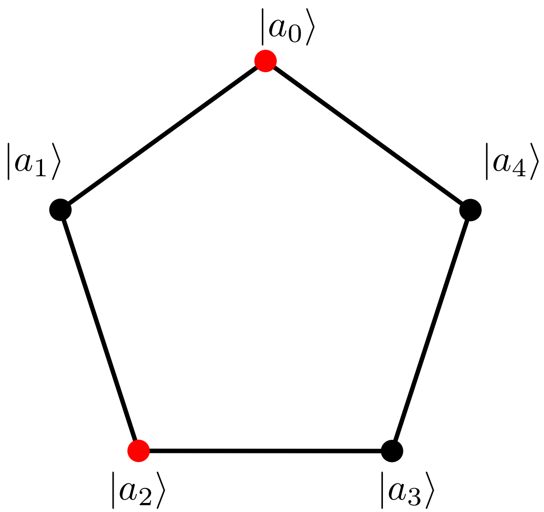
The model is required to reproduce the quantum predictions, i.e.

$$\int_\Lambda \text{Pr}(a|M, \lambda) d\mu_\psi = |\langle a|\psi\rangle|^2 .$$

Example: Klyachko inequality

- Ususal proofs of contextuality inequalities use $\Gamma_a \cap \Gamma_b = \emptyset$ when $|\langle a|b\rangle|^2 = 0$.
- Example:
  - $|a_j\rangle = \sin\vartheta\cos\varphi_j\,|0\rangle + \sin\vartheta\sin\varphi_j\,|1\rangle + \cos\vartheta\,|2\rangle$
  - $\varphi_j = \frac{4\pi j}{5}$ and $\cos\vartheta = \frac{1}{\sqrt[4]{5}}$





$$\langle L_c\rangle = \frac{1}{5}\sum_{j=0}^{4} L_c(a_j,\psi) \leq \frac{2}{5}$$

- For a (finite) set V of states, a noncontextuality inequality is a bound of the form

$$\sum_{|a\rangle \in V} \mu_\psi(\Gamma_a) \leq \gamma.$$

- Let $\mathcal{M}$ be a covering set of bases for V. We have

$$\int_\Lambda \text{Pr}(a|M,\lambda) d\mu_a(\lambda) = |\langle a|a\rangle|^2 = 1$$

and since $\text{Pr}(a|M,\lambda) \leq 1$ this implies that $\mu_a(\Gamma_a^M) = 1$.

- Since $\Gamma_a = \cap_{M \in \mathcal{M} | |a\rangle \in M} \Gamma_a^M$ is a finite intersection of measure one sets, we also have

$$\mu_a(\Gamma_a) = 1.$$

- Now,

$$L_c(\psi, a) = \inf_{\Omega \in \Sigma} [\mu_\psi(\Omega) + \mu_a(\Lambda \backslash \Omega)]$$
$$\leq \mu_\psi(\Gamma_a) + \mu_a(\Lambda \backslash \Gamma_a)$$

- We just showed that $\mu_a(\Gamma_a) = 1$, so $\mu_a(\Lambda \backslash \Gamma_a) = 0$, and hence

$$L_c(\psi, a) \leq \mu_\psi(\Gamma_a).$$

- Hence,

$$\sum_{|a\rangle \in V} L_c(\psi, a) \leq \sum_{|a\rangle \in V} \mu_\psi(\Gamma_a) \leq \gamma.$$

- Using Cabello, Severini and Winter's results[6], for a set of states V, we can derive

$$\frac{1}{|V|}\sum_{|a\rangle\in V} L_c(\psi,a) \leq \frac{\alpha(G)}{|V|},$$

where $\alpha(G)$ is the *independence number* of the *orthogonality graph* of V.

- Better bounds come from a different technique, introduced by Barrett et. al.[7], that was not based on contextuality.

- It turns out that their method is contextuality in disguise though.

[6] A. Cabello, S. Severini, A. Winter, Phys. Rev. Lett. 112:040401 (2014).
[7] J. Barrrett et. al., Phys. Rev. Lett. 112, 250403 (2014)

Antidistinguishability

- Definition: A set $V = \{|a_j\rangle\}_{j=1}^d$ of states in $\mathbb{C}^d$ is *antidistinguishable* if there exists an orthonormal basis $\left\{\left|a_j^\perp\right\rangle\right\}_{j=1}^d$ such that, for all j,

$$\left|\left\langle a_j^\perp \middle| a_j \right\rangle\right|^2 = 0.$$

- Example:

$$|a_1\rangle = (1,0,0) \qquad \left|a_1^\perp\right\rangle = (0,1,0)$$

$$|a_2\rangle = (1,1,1) \qquad \left|a_2^\perp\right\rangle = (1,0,-1)$$

$$|a_3\rangle = (-1,1,1) \qquad \left|a_3^\perp\right\rangle = (1,0,1)$$

Implication for ontological models

- Theorem: If V is antidistinguishable then

$$\cap_{j=1}^{d} \Gamma_{a_j} = \emptyset.$$

- Proof: Because ontic states in $\cap_{j=1}^{d} \Gamma_{a_j}$ would have to assign probability 0 to all of the measurement outcomes $\left| a_j^{\perp} \right\rangle$.

Bonferroni inequalities

- On any measure space, the inclusion-exclusion principle states:

$$\mu(\cup_j X_j) = \sum_j \mu(X_j) - \sum_{j<k} \mu(X_j \cap X_k) + \sum_{j<k<m} \mu(X_j \cap X_k \cap X_m) - \dots$$

- Bonferroni: Terminating this sequence gives an alternating sequence of upper and lower bounds, e.g.

$$\mu(\cup_j X_j) \leq \sum_j \mu(X_j)$$

$$\mu(\cup_j X_j) \geq \sum_j \mu(X_j) - \sum_{j<k} \mu(X_j \cap X_k).$$

- Set $X_j = \Gamma_\psi \cap \Gamma_{a_j}$ and note that $\mu_\psi(\Gamma_\psi) = 1$. Second inequality gives

$$1 \geq \sum_j \mu_\psi(\Gamma_{a_j}) - \sum_{j<k} \mu(\Gamma_\psi \cap \Gamma_{a_j} \cap \Gamma_{a_k})$$

- From previous slide:

$$1 \geq \sum_j \mu_\psi(\Gamma_{a_j}) - \sum_{j<k} \mu(\Gamma_\psi \cap \Gamma_{a_j} \cap \Gamma_{a_k})$$

- So, if $\{|\psi\rangle, |a_j\rangle, |a_k\rangle\}$ are antidistinguishable for all $j \neq k$, we get

$$\sum_j \mu_\psi(\Gamma_{a_j}) \leq 1.$$

- Example: Yu-Oh inequality[8]

$$|\psi\rangle = (1,0,0)$$

$$|a_0\rangle = \frac{1}{\sqrt{3}}(1,1,1) \qquad |a_1\rangle = \frac{1}{\sqrt{3}}(-1,1,1)$$

$$|a_2\rangle = \frac{1}{\sqrt{3}}(1,-1,1) \qquad |a_3\rangle = \frac{1}{\sqrt{3}}(1,1,-1)$$

[8] S. Yu, C. Oh, Phys. Rev. Lett. 108, 030402 (2012)

- Let,

$$|\psi\rangle = (1, 0, 0, \cdots, 0).$$

- For $\boldsymbol{x} \in \{0,1\}^d$, let

$$|a_{\boldsymbol{x}}\rangle = (-1^{x_1}, -1^{x_2}, \ldots, -1^{x_n}).$$

- Then, $\{|\psi\rangle, |a_{\boldsymbol{x}}\rangle, |a_{\boldsymbol{x}'}\rangle\}$ is antidistinguishable for $\boldsymbol{x} \neq \boldsymbol{x}'$, so

$$\sum_{\boldsymbol{x}} \mu_\psi(\Gamma_{a_{\boldsymbol{x}}}) \leq 1$$

- In contrast, using CSW method on this set only gives

$$\sum_{\boldsymbol{x}} \mu_\psi(\Gamma_{a_{\boldsymbol{x}}}) \leq (2-\epsilon)^d$$

for some $\epsilon > 0$.