# Weak values, Local Momentum and Structure Process.


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http://www.hep.ucl.ac.uk/qupot/.

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## EMERGENT SPACE-TIME and CLASSICAL PHYSICS.

Take quantum physics as basic.

NO! It is about **PROCESS, ACTIVITY**, about **BECOMING.**

[Bohm. Space, Time, and Q.T. Understood in Terms of Discrete Structural Process, Proc. Int. Conf.' Kyoto, 252-287, (1965).]

1. To explain and develop the mathematics we need to describe **STRUCTURE PROCESS.**

Artist's impression of structure process.

This sculpture can be seen on the South Bank of the river Thames in London

The Quantum Cloud. A. Gormley.

**EMERGENT SPACE-TIME and CLASSICAL PHYSICS.**

Take quantum physics as basic. Is it about material objects moving in space and time?

NO! It is about **PROCESS, ACTIVITY**, about **BECOMING.**

[Bohm. Space, Time, and Q.T. Understood in Terms of Discrete Structural Process, Proc. Int. Conf.' Kyoto, 252-287, (1965).]

1. To explain and develop the mathematics we need to describe **STRUCTURE PROCESS.**

We need an **ALGEBRAIC** description of quantum phenomena.

Orthogonal and symplectic Clifford algebra. Geometric algebras

[Hiley,Process, Distinction, Groupoids and Clifford Algebras, Lecture Notes in Physics, vol. 813, 705, Springer (2011)]

2. I will show how the standard quantum formalism and classical mechanics emerge from this structure.

[de Gosson and Hiley, Found Physics, **41** (2011), 1415-1436.]

[de Gosson and Hiley, Phys. Letts. **A 377** (2013) 3005–3008.]

3. Finally I want to focus on the **weak value** of the **momentum operator**.

Also known as the (i) **local momentum**, (ii) Bohm momentum, (iii) guidance condition.

[Berry, Five Momenta, Eur. J. Phys. **34**, (2013) 1337–1348.]

I show how the weak values fit comfortably into a **non-commutative phase space**.

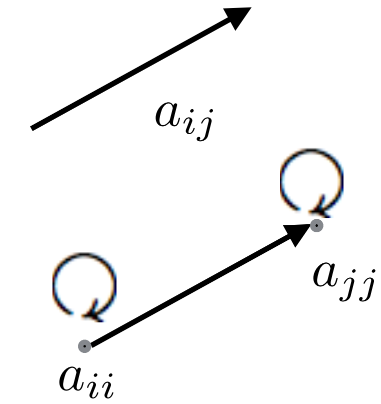
[Hiley, J. Phys. Conf. Series, **361** (2012) 012014.]

[Hiley, J. of Comp. Electronics, **14** (2015) 869-878.]

The aim of our experimental group is to explore this structure in phase space.

# What do we mean by STRUCTURE PROCESS?

Rather than taking BEING as primary, we take BECOMING as primary




BEING is a special case of becoming. 

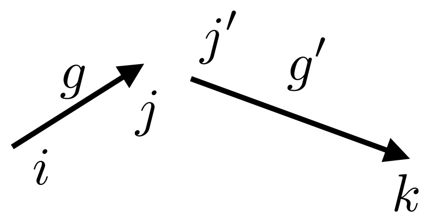
It is an IDEMPOTENT $a_{ii}a_{ii} = a_{ii}$

**Eddington:** What is a ‘particle’?

“The structural concept of existence is represented by an idempotent symbol”

[Eddington, The Philosophy of Physical Science, p. 162.]

The structure process forms a **groupoid.** Order of succession defined by the product:-




$g, g' \in G$ and $j = j'$

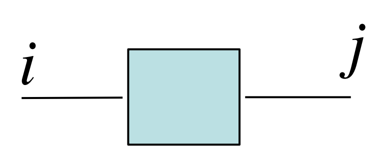
$$a(nm) = \sum_{k=0}^{\infty} b(nk)c(km)$$

$j$ is the target; $j$’ is the source.

[Landsman, Mathematical Topics between Classical and Quantum Mechanics,1998.]

Order the elements of the groupoid to form an matrix algebra.

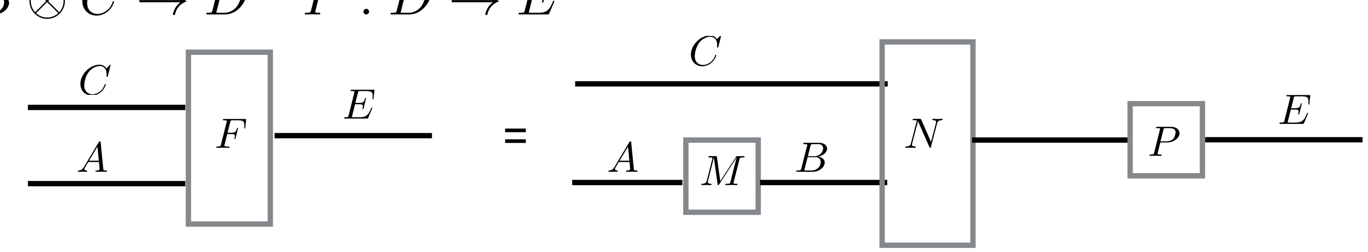
$$A \to \begin{pmatrix} a_{11} & . & . & . & . \\ . & . & . & a_{lm} & . \\ . & . & a_{ii} & . & . \\ . & . & . & . & . \\ . & . & . & . & a_{nn} \end{pmatrix} \Rightarrow$$




Heisenberg matrix mechanics is a primitive precursor.

All of this can be made rigorous in terms of a **monoidal category**

$$F_m^{ik} = P_m^l N_l^{jk} M_j^i \quad \Rightarrow \quad M : A \to B \quad N : B \otimes C \to D \quad P : D \to E$$




[Abramsky and Coecke, quant-ph/0402130]

## Heisenberg-Born-Jordan Quantum Mechanics.

[Heisenberg, Z. Phys. **33** (1925), 879–893.]

[Born, Jordan, Z. Phys, **34** (1925) 858-888.]

By writing $q \to q(nm) \to a(nm)\exp[2\pi\nu(nm)t$

Heisenberg explains the Ritz-Rydberg combination principle by writing: $q(nm) = q(nk)q(km)$

viz:- $\exp[2\pi i\nu(nm)] = \exp[2\pi i\nu(nk)t] \times \exp[2\pi i\nu(km)t] \Rightarrow \boxed{\nu(nm) = \nu(nk) + \nu(km)}$

In terms of Bohr's energy levels:-

$$h\nu(nm) = (E_n - E_m) = (E_n - E_k) + (E_k - E_m) = h[\nu(nk) + \nu(km)]$$

We can go further and write:- $E(mn) = E_m\delta_{mn}$ so that

$$Eq = \sum_k E(mk)q(kn) = \sum_k E_m\delta_{mk}q(kn) = E_m q(mn)$$

$$qE = \sum_k q(mk)E(kn) = \sum_k q(mk)E_k\delta_{kn} = E_n q(mn)$$

Thus $Eq - qE = (E_m - E_n)q(mn)$

Non-commutative structure

$$\dot{q} = \frac{i}{\hbar}(Eq - qE)$$

Essentially Heisenberg eqn of motion.

NB. No Schrödinger equation

Notice we have written

$$q \to q(mn)\exp[2\pi i\nu(mn)t]$$

Phase angle

**We need something more general.**

**Polar decomposition of a matrix**

For any $A \in GL(n, \mathbb{C})$ there exist a unique unitary matrix U and positive definite matrices $P_1, P_2$ such that

$A = UP_1 = P_2U$, where $P_1 = U^\dagger P_2 U$ and $U = \exp[iS]$ with $S^\dagger = S$

**This enables us to handle the Pauli and Dirac spinors.**

# Wave Function information is encoded in a matrix.

**Pauli**

$$\psi_L = g_0 + g_1\sigma_{23} + g_2\sigma_{13} + g_3\sigma_{12}$$
$$= \begin{pmatrix} g_0 + ig_3 & ig_1 - g_2 \\ ig_1 + g_2 & g_0 - ig_3 \end{pmatrix}$$

$$\begin{cases} g_0 = (\psi_1^* + \psi_1)/2 & g_3 = i(\psi_1^* - \psi_1)/2 \\ g_2 = (\psi_2^* + \psi_2)/2 & g_1 = i(\psi_2^* - \psi_2)/2 \end{cases}$$

$g_0, g_1, g_2, g_3 \in \mathbb{R}$

**Dirac**

$$\psi_L = a + b\gamma_{12} + c\gamma_{23} + d\gamma_{13} + f\gamma_{01} + g\gamma_{02} + h\gamma_{03} + n\gamma_5 \longleftarrow \text{4 x 4 matrix}$$

$$\psi_1 = a - ib; \quad \psi_2 = -d - ic; \quad \psi_3 = h - in; \quad \psi_4 = f + ig$$

$a, b, c, d, f, g, h, n \in \mathbb{R}$

**Schrödinger**

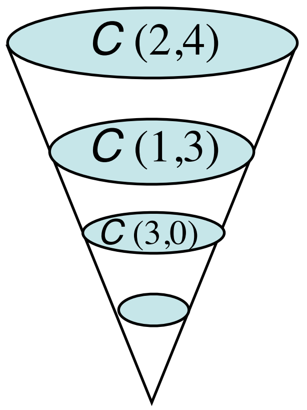
$$\psi_L = g_0 + eg_1 = \begin{pmatrix} g_0 & g_1 \\ -g_1 & g_0 \end{pmatrix}$$

$$g_0 = (\psi^* + \psi)/2 \qquad g_1 = i(\psi^* - \psi)/2 \qquad g_0, g_1 \in \mathbb{R}$$

Instead of

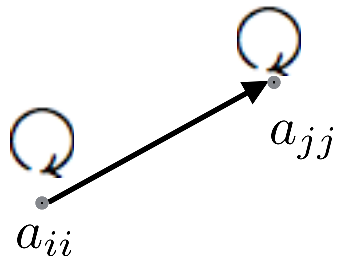
$$\Psi \ , \qquad \Psi = \begin{pmatrix} R_1 e^{iS_1} \\ R_2 e^{iS_2} \end{pmatrix} \qquad , \qquad \Psi = \begin{pmatrix} R_1 e^{iS_1} \\ R_2 e^{iS_2} \\ R_3 e^{iS_3} \\ R_4 e^{iS_4} \end{pmatrix}$$

Clifford Hierarchy




[Hiley,Process, Distinction, Groupoids and Clifford Algebras, Lecture Notes in Physics, vol. 813, 705, Springer (2011)]

## Dirac's bra-ket notation already contains the notion of process.




$|i\rangle\langle j|$ Outer product

$|i\rangle\langle i| = \epsilon_i$ Idempotent since $|i\rangle\langle i|\,|i\rangle\langle i| = |i\rangle\langle i|$

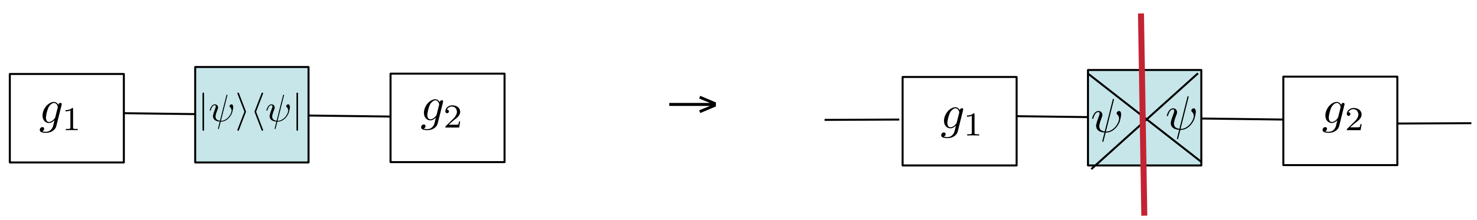
Inner product $\langle i|j\rangle$ This is the transition probability amplitude of finding process $i$, given process $j$

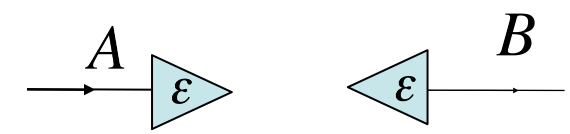
post-selection pre-selection

The wave function is a transition probability amplitude!

Given the process labeled by $|\psi\rangle$, the probability amplitude of finding the energy at $x$ is $\langle x|\psi\rangle$

We need to find a better, algebraic way of describing the individual process.




**$\in$ left ideal** **$\in$ right ideal**

$\Psi_L = A\varepsilon$ $\Psi_R = \varepsilon B$

$$\rho(i,j) = \langle i|\psi\rangle\langle\psi|j\rangle = \Psi_L(i)\Psi_R(j) = \psi_L(i)\epsilon\psi_R(j)$$

$\Psi_L$ is a element of a minimal left ideal

$\Psi_R$ is a dual right ideal

Now we can include spin

For the Pauli $\epsilon_P = (1+\sigma_3)/2$ Picks a spatial axis

Orthogonal Clifford algebra

For the Dirac $\epsilon_D = (1+\gamma_0)/2$ Picks the time axis

[Hiley and Callaghan, Foundations of Physics, 42 (2012) 192-208.]

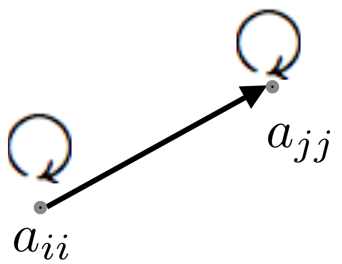
For the Schrödinger $\epsilon_S = 1$ $\epsilon = \rangle\langle$ Symplectic Clifford algebra

[Dirac, The Principles of Quantum Mechanics, 1947]

[Hiley, Found. Phys. 40, (2010) 356-367.]

# How do the statistics come in?

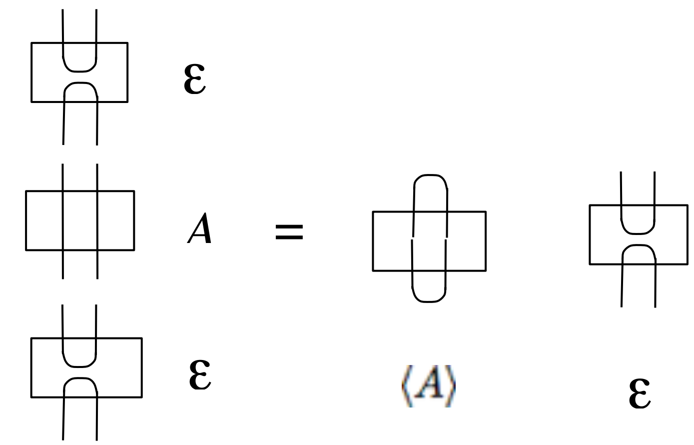
Arises naturally from Wederburn's theorem for a simple algebra




$$\epsilon A \epsilon = \langle A \rangle \epsilon$$

$$\epsilon^2 = \epsilon$$

Expectation value



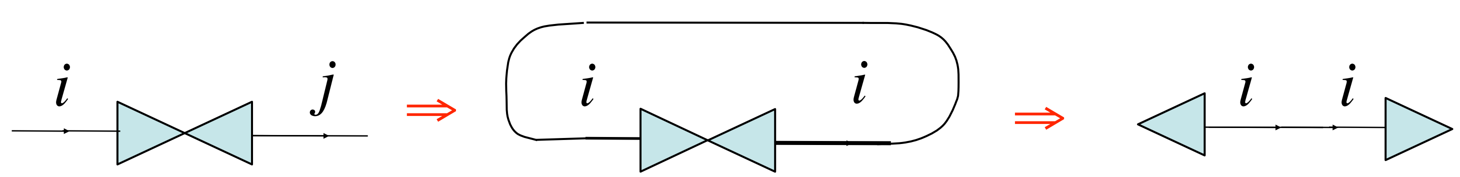

[Kauffman, Knots and Physics, p. 377 (2001)]

If we write $\epsilon = |\psi\rangle\langle\psi|$

$$|\psi\rangle\langle\psi|A|\psi\rangle\langle\psi| = \langle A \rangle |\psi\rangle\langle\psi|$$

We can also write $\langle A \rangle = Tr(\rho A)$ where $\rho = |\psi\rangle\langle\psi| = \Psi_L \Psi_R = \psi_L \epsilon \psi_R$

**Full algebraic description of quantum phenomena**




**Operator** **Trace** **Complex number**

## How do weak values fit into all this algebra?

$$\langle\psi|A|\psi\rangle = tr(A\rho) \qquad \text{with} \qquad \rho = |\psi\rangle\langle\psi| \qquad \hat{A}|a_n\rangle = a_n|a_n\rangle \qquad a_{\text{min}} \leq a_n \leq a_{\text{max}}.$$

$$tr(A\rho) = \sum_{\phi_j} \rho(\phi_j)\frac{\langle\phi_j|A|a_n\rangle}{\langle\phi_j|\psi\rangle} = \sum_{\phi_j} \rho(\phi_j)\langle A_{(\phi_j;\psi)}\rangle_w \qquad \rho(\phi_j) = |\langle\phi_j|\psi\rangle|^2$$

Weak Value $\langle A_{(\phi_j;\psi)}\rangle_w = \dfrac{\langle\phi_j|\hat{A}|\psi\rangle}{\langle\phi_j|\psi\rangle}$ **Transition probability amplitude**

$$\boxed{|\phi_j\rangle = \sum_{j,n} d_{jn}|a_n\rangle} \text{ and } \quad |\psi\rangle = \sum_k c_k|a_k\rangle.$$

$$\langle A_{(\phi_j;\psi)}\rangle_w = \frac{\langle\phi_j|\hat{A}|\psi\rangle}{\langle\phi_j|\psi\rangle} = \frac{\sum_{n,k} d^*_{jn}c_k\langle a_n|\hat{A}|a_k\rangle}{\sum_{n.k} d^*_{jn}c_k\langle a_n|a_k\rangle} = \frac{\sum_n d^*_{jn}c_n a_n}{\sum_n d^*_{jn}c_n}.$$

Two state system we see that with these coefficients:-

$$a_1 = -1,\ a_2 = 1 \qquad d_{j1} = -99 \qquad d_{j2} = 101 \qquad c_1 = c_2 = 1 \qquad \langle A_{(\phi_j;\psi)}\rangle_w = \frac{\langle\phi_j|\hat{A}|\psi\rangle}{\langle\phi_j|\psi\rangle} = 100$$

**A weak value is a weighted transition probability amplitude,** $A(\phi_j\ \psi)$

The standard approach arises by choosing

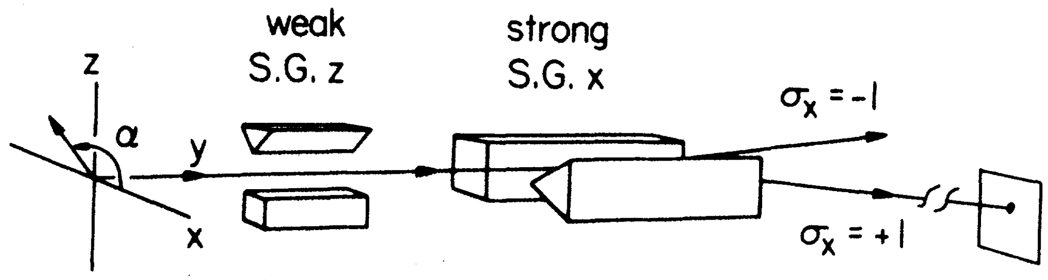
$$\boxed{|\phi_j\rangle = |a_j\rangle} \text{ and } \quad |\psi\rangle = \sum_k c_k|a_k\rangle$$

$$\rho(a_j) = |c_j|^2$$

$$\langle A_{(\phi_j;\psi)}\rangle_w = \frac{\sum_k c_k\langle a_j|\hat{A}|a_k\rangle}{\sum_k c_k\langle a_j|a_k\rangle} = \frac{c_j a_j}{c_j} = a_j.$$

$$\langle\psi|A|\psi\rangle = \sum_j |c_j|^2 a_j$$

# Weak values of spin.




[Duck, Stevenson and Sudarshan, Phys. Rev. **40** (1989) 2112-17.]

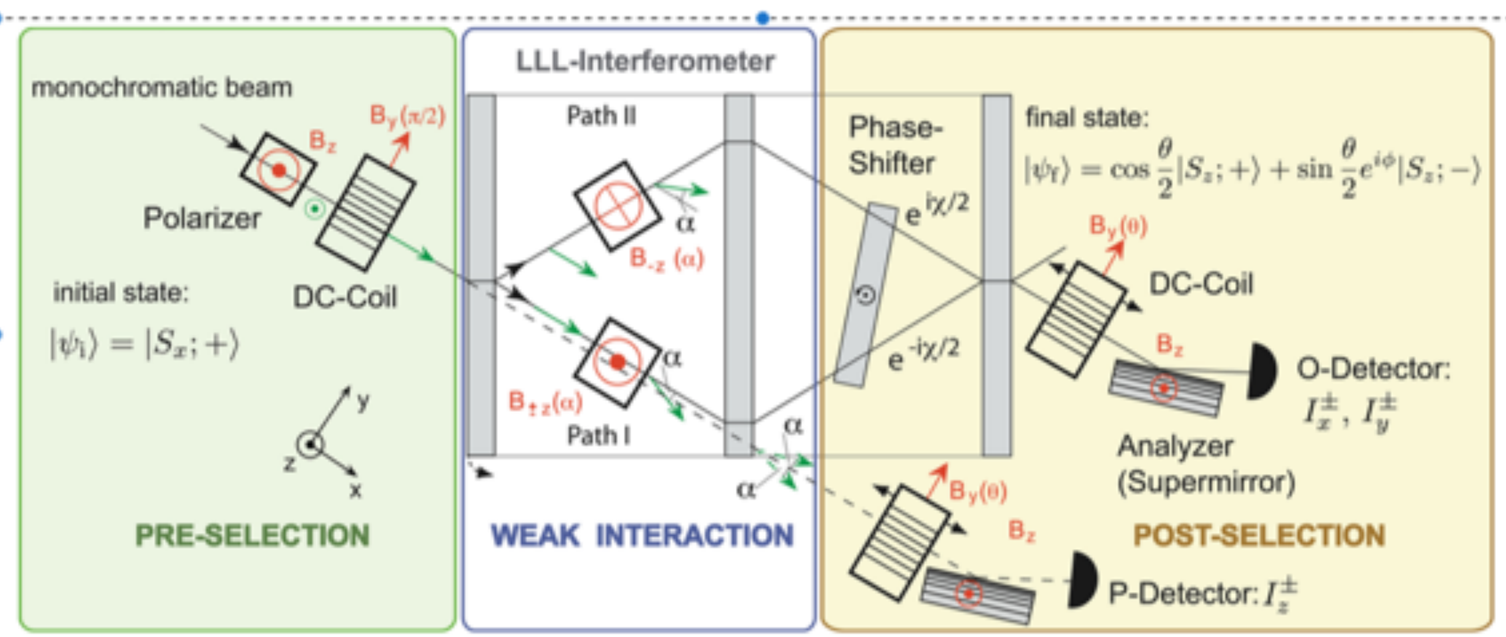
$$\langle \hat{\sigma}_{z(\phi;\psi)} \rangle_w = \frac{\langle \phi | \hat{\sigma}_z | \psi \rangle}{\langle \phi | \psi \rangle}$$

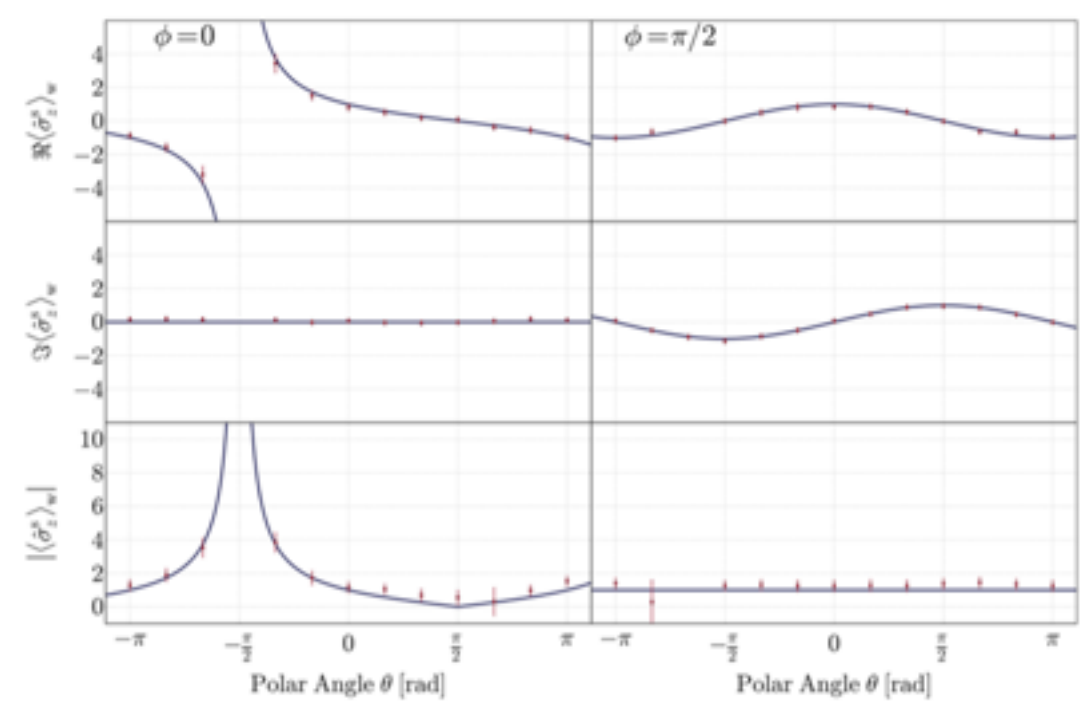
$$\langle \phi | = \langle \uparrow_z | + \langle \downarrow_z | \quad \text{and} \quad \hat{\sigma}_z | \psi \rangle = d_+ | \uparrow_z \rangle + d_- | \downarrow_z \rangle \rangle.$$

$$\sqrt{2} d_+ = \cos \alpha/2 + \sin \alpha/2 \quad \text{and} \quad \sqrt{2} d_- = \cos \alpha/2 - \sin \alpha/2,$$

$$\langle \hat{\sigma}_{z(\phi;\psi)} \rangle_w = \tan \alpha/2.$$

## Experiment using neutrons.




[Sponar, Denkmayr,,Geppert, Lemmel, Matzkin, and Hasegawa, arXiv:1404.2125 (2014).]

**Rob Flack and Vincenzo Monachello are experimentally investigating weak values of spin using helium.**

# Weak value of the momentum operator, or the LOCAL MOMENTUM.

**Notion of a local momentum has a long history**

$$\langle P_{(x,\psi)}\rangle_w = \frac{\hat{P}\psi(x,t)}{\psi(x,t)} = p(x,t) = m(v+iv_i) \qquad \psi = R\exp[iS/\hbar] \qquad \rho = R^2$$

$$v = \frac{\nabla S}{m} \qquad v_i = -\frac{\hbar\nabla ln\rho}{2m} = -\frac{\hbar}{m}\frac{\nabla R}{R}$$

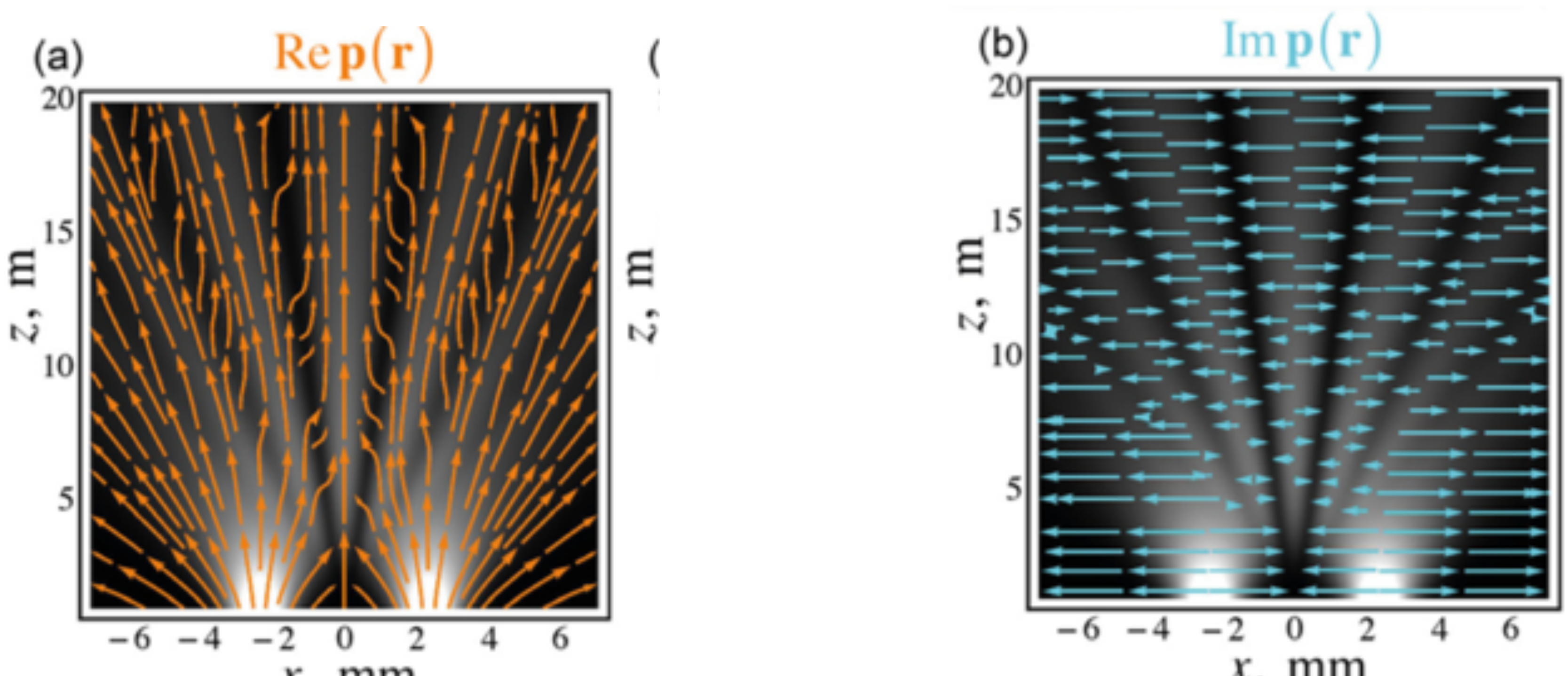
Osmotic velocity

[London, Rev. Mod. Phys. 17 (1945) 310-20]

[Hirschfelder, Christoph and Palke, J. Chem Phys. 61 (1974) 5435-55].

[Nelson, Dynamical Theories of Brownian Motion, (1967),]

[Berry, Five Momenta, Eur. J. Phys. **34**, (2013) 1337–1348.]




[Bliokh, Bekshaev, Kofman and Nori, New J. Phys. 15 (2013) 073022 ]

## The weak value of the kinetic energy:-

$$(2m\psi)^{-1}\hat{\boldsymbol{p}}.\hat{\boldsymbol{p}}\psi = (2m\psi)^{-1}\hat{\boldsymbol{p}}.(\boldsymbol{v}+i\boldsymbol{v}_i)\psi = mv^2/2 + Q + i(m\boldsymbol{v}.\boldsymbol{v}_i - \hbar\nabla.\boldsymbol{v})$$

$$2Q = -mv_i^2 + \hbar\nabla.\boldsymbol{v}_i = -\frac{\hbar^2}{m}\frac{\nabla^2 R}{R}$$

This gives us a very different view of the quantum potential energy

[Grössing, Physica A: Statistical Mechanics and its Applications **388.**6 (2009): 811-823.]

# Stationary states

**Consider the ground state of particle in box.**

$$\psi_n(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)e^{\frac{iE_n t}{\hbar}} \qquad R_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right). \qquad S_n(x,t) = \frac{E_n t}{\hbar}$$

$$\partial_x S_n = 0 \quad \text{and} \quad \partial_t S_n = -E_n \qquad Q_n = -\frac{\hbar^2}{2m}\nabla^2 R_n/R_n = -\frac{n^2\hbar^2\pi^2}{2ma}$$

This means $\boldsymbol{v} = 0$ but $\boldsymbol{v}_i \neq 0$

[Bohm and Hiley, Phys. Rev. Letters **55**, (1985) 2511-1514.]

**Why don't we see the effects of $v_i$ in standard QM?**

$$\langle\psi|\hat{P}|\psi\rangle = \hbar\int\left[\rho\nabla S + \frac{1}{2}\nabla\rho\right]d^3x = \hbar\int\left[\rho\nabla S + \frac{\rho}{2}\frac{\nabla\rho}{\rho}\right]d^3x = \frac{\hbar}{m}\int\left[\rho\boldsymbol{v} + \frac{\rho}{2}\boldsymbol{v}_i\right]d^3x.$$

Since $\rho \to 0$ as $x \to \pm\infty$ then $\int \nabla\rho\, d^3x = 0$

Therefore only the local velocity contributes to the mean momentum.

However as we have seen the **weak value of the kinetic energy is**

$$\langle P^2_{x,\psi}(x,t)\rangle_w/2m = mv^2/2 + Q$$

$$\langle P_{x,\psi}\rangle_w = T^{0j}/\rho = mv^j$$

$$\langle\psi|\hat{P}^2|\psi\rangle = \int\rho[(mv)^2 + 2mQ]d^3x.$$

# How does classical mechanics emerge?

Separate the real and imaginary parts of the Schrödinger equation under polar decomposition

$$\psi(x,t) = R(x,t)\exp[iS(x,t)]$$

**Imaginary part**

$$\frac{\partial P_x}{\partial t} + \nabla_x . \left(P_x \frac{\nabla_x S_x}{m}\right) = 0 \qquad \text{with} \quad P_x = |\psi(x)|^2 \qquad \text{Conservation of probability}$$

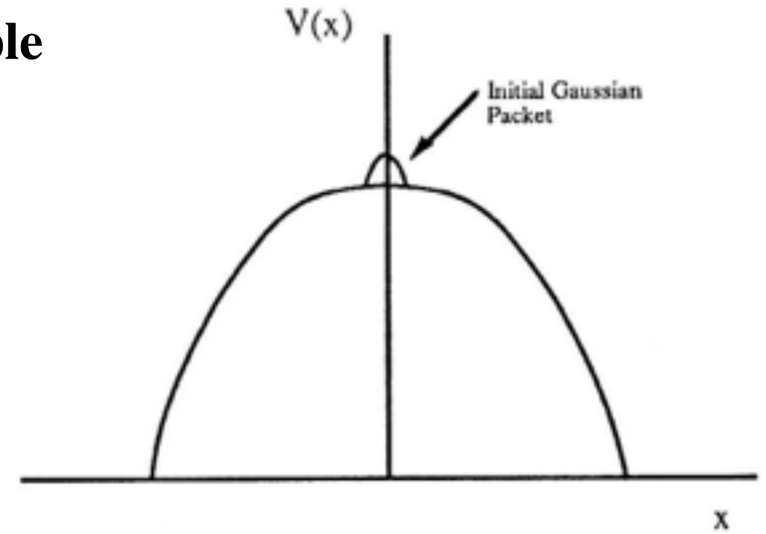
**Real part**

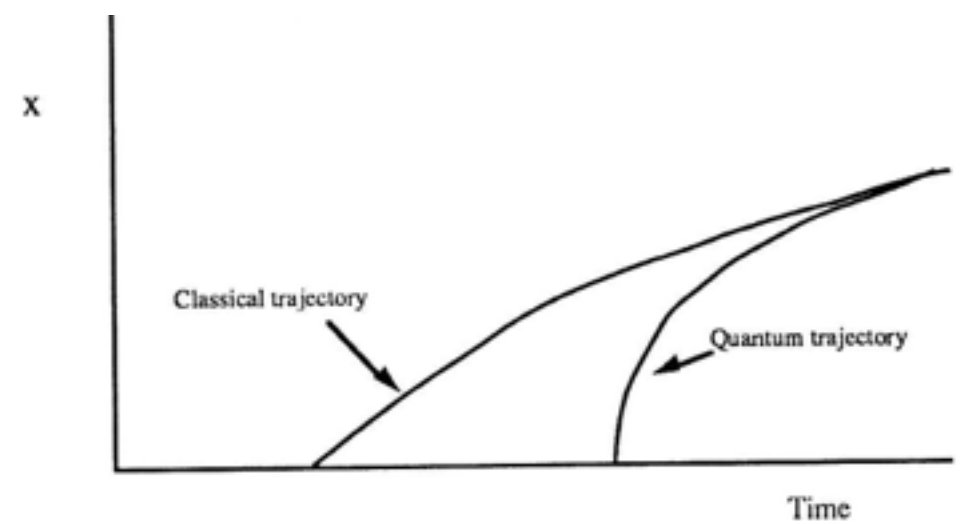
$$\frac{\partial S_x}{\partial t} + \frac{1}{2\pi}\left(\frac{\partial S_x}{\partial x}\right) - \frac{1}{2mR_x}\left(\frac{\partial^2 R_x}{\partial x^2}\right) + V(x) = 0 \qquad \text{Quantum Hamilton-Jacobi equation}$$

Quantum potential $Q(x,t)$

Classical mechanics emerges when $Q$ becomes negligible.

**Example**




[Hiley and Mufti, in Fundamental Problems in Quantum Physics, ed. Ferrero and van der Merwe, pp.141-156, Kluwer, Dordrecht. (1995)

Suggests it is *not inconsistent* to regard this as a ‘trajectory’ but beware!

Novel flow of energy. $\quad \frac{1}{2m}(\nabla S)^2 \leftrightarrow \frac{1}{2m}\left[(\nabla S)^2 - \hbar^2\frac{\nabla^2 R}{R}\right]$

# Local Momentum.

As I have already remarked, the notion of LOCAL MOMENTUM has been around a long time.

It was discussed by Hirschfelder, Christoph and Palke, J. Chem Phys. 61 (1974) 5435-55.

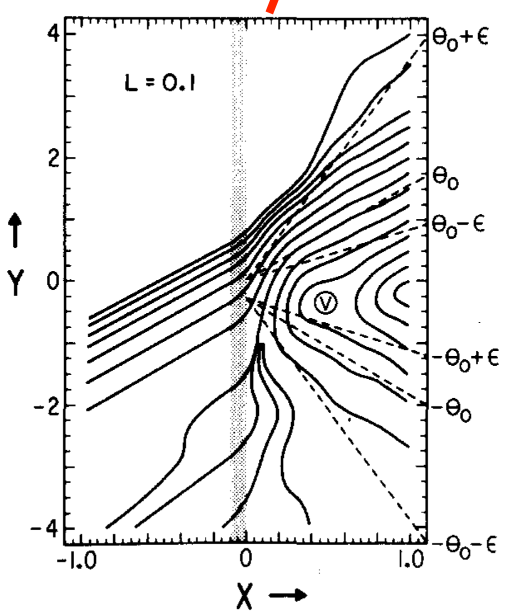
Belinfante, A Survey of Hidden-Variables Theories, 1973, who called it the "Bohm momentum."

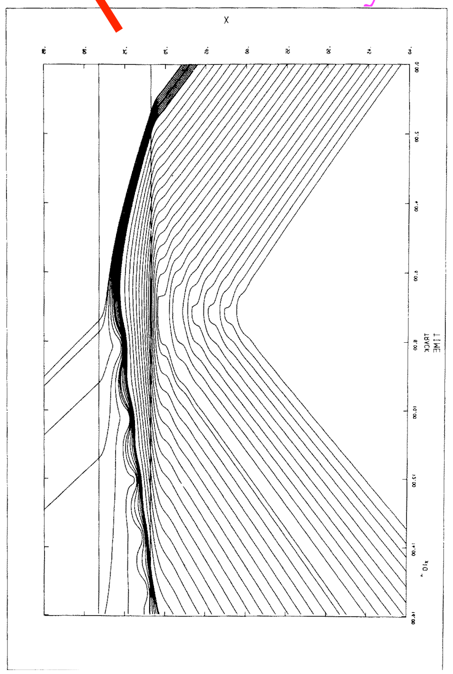
Philippidis, Dewdney and Hiley, Nuovo Cimento 52B, 15-28 (1979).

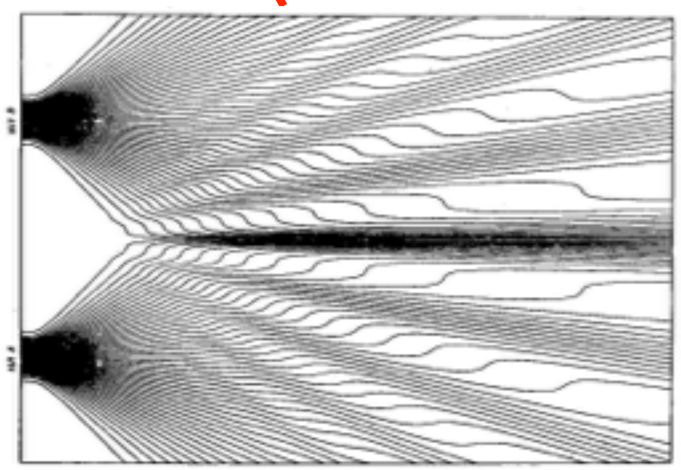
Dewdney and Hiley Found. Phys. 12, 27-48 (1982).

Bohm and Hiley, The Undivided Universe, 1993.

Holland, The quantum theory of motion, 1995.





**Why was all this work been ignored for so long?**

Fritz London in Rev. Mod. Phys., **17**, (1945) 310.

"The local mean velocity has no true quantum mechanical significance since it cannot be expressed as the expectation value of *any* linear operator."

What has changed? Steinberg's group has shown we can measure local momenta using

**Peter Barker, Pete Edmunds and Joel Morley are attempting the same thing using argon**