# Foundation of Quantum Mechnics studied in Matter-Wave optics
## Quantum Cheshire-Cat and Uncertainty Relations

**Yuji HASEGAWA**
Atominsitut, TU-Wien, Vienna, AUSTRIA

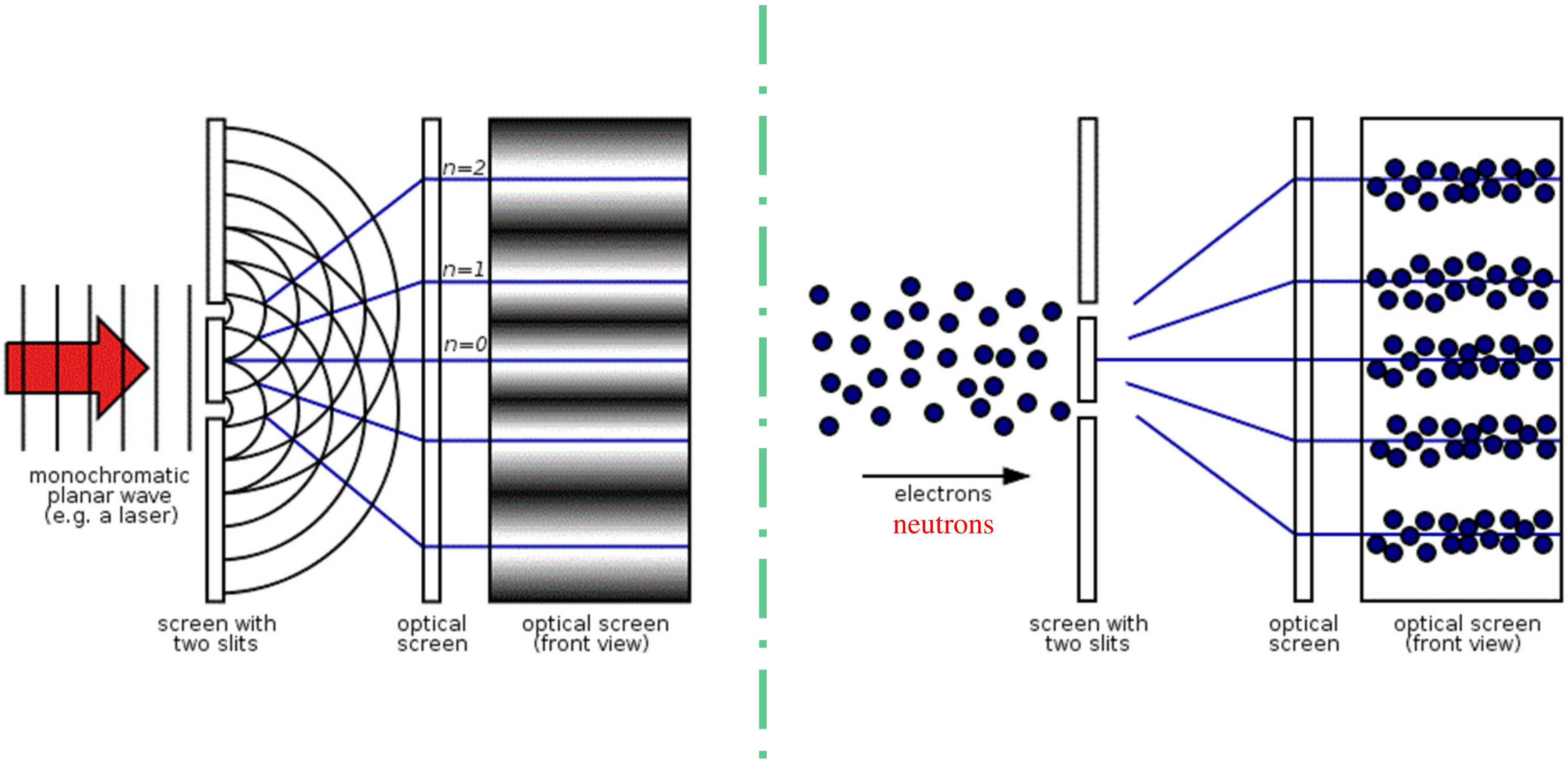
**I. Introduction: neutron as a particle/wave**

**II. Quantum Cheshire-cat & weak measurement of spin-1/2**

**II. Uncertainty relations for error/noise-disturbance**

**V. Summary**

# Waves/Nonlocality in classical- and quantum



# Neutron interferometry

**Neutrons**

$m = 1.67\times 10^{-27}$ kg

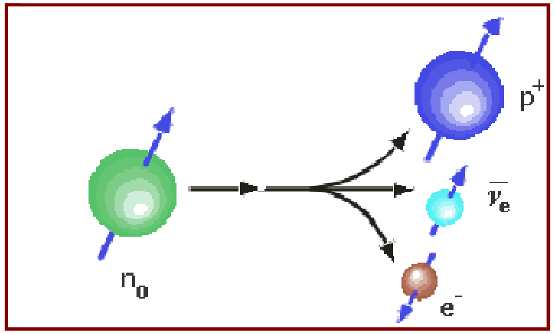
$s = \frac{1}{2}\hbar$

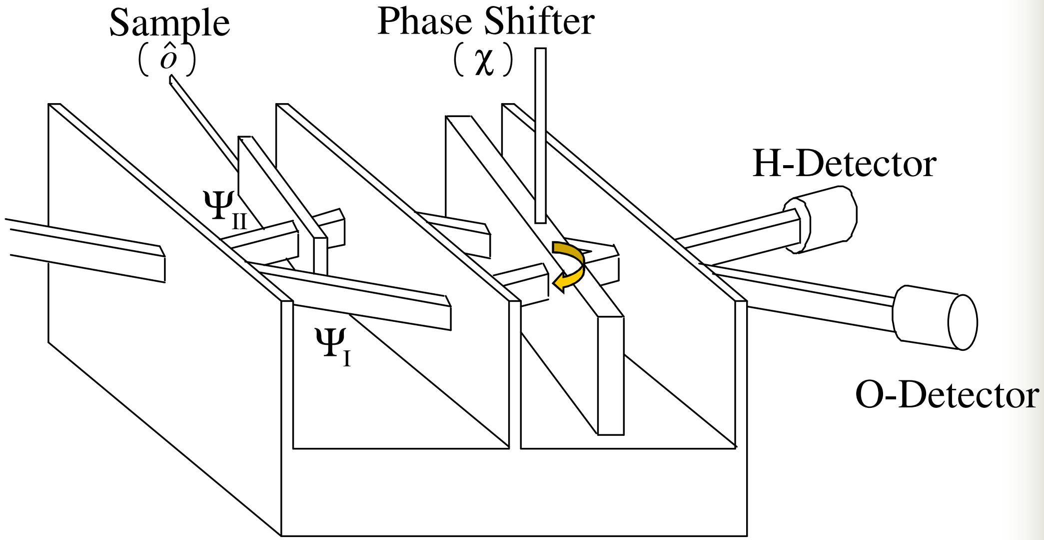
$\mu = -9.66\times 10^{-27}$ J/T

$\tau = 887$ s

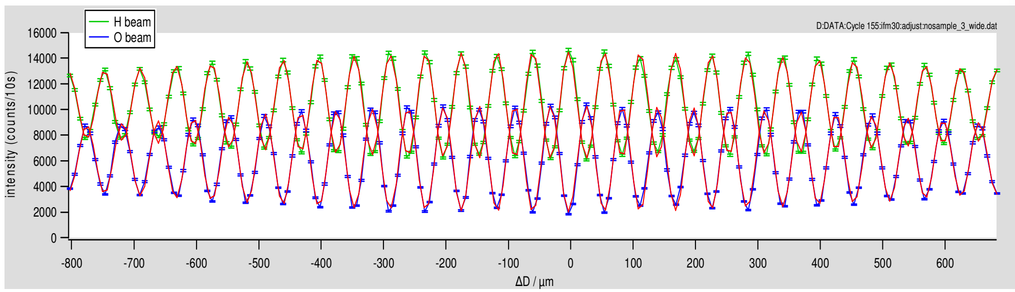
$R = 0.7$ fm

u–d–d quark structure

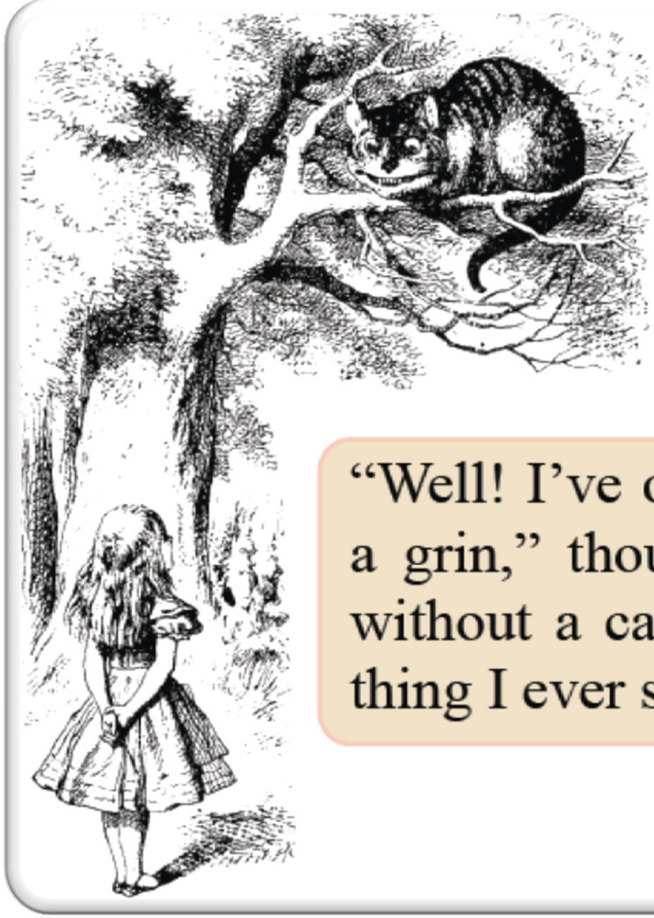



$$I = |\Psi_I + e^{i\chi}\cdot\hat{o}\cdot\Psi_{II}|^2$$

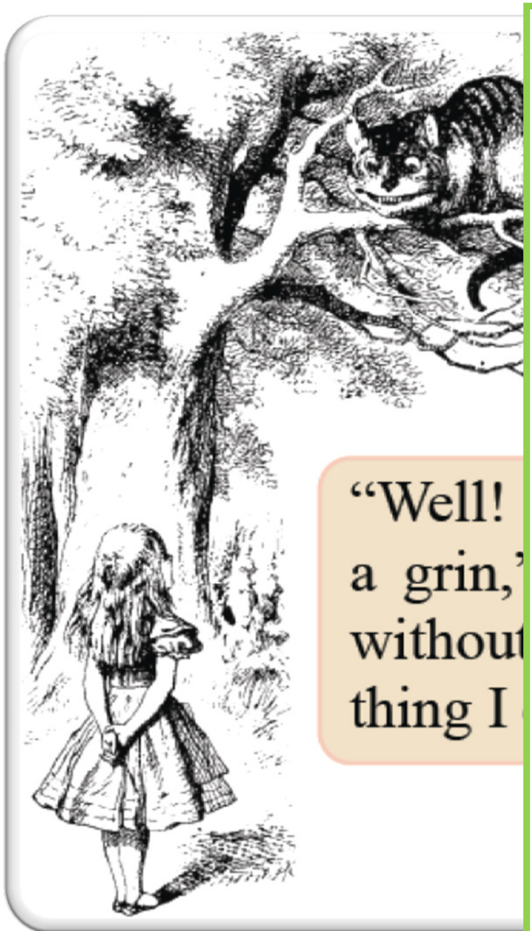


# Cheshire-cat



"Well! I've often seen a cat without a grin," thought Alice; "but a grin without a cat! It's the most curious thing I ever saw in all my life!"

# Quantum Cheshire-cat




**New Journal of Physics**
The open access journal for physics

## Quantum Cheshire Cats

**Yakir Aharonov[1,2], Sandu Popescu[3,6], Daniel Rohrlich[4,6] and Paul Skrzypczyk[5]**

[1] Tel Aviv University, School of Physics and Astronomy, Tel Aviv 69978, Israel
[2] Schmid College of Science, Chapman University, 1 University Drive, Orange, CA 92866, USA
[3] H. H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol, BS8 1TL, United Kingdom
[4] Physics Department, Ben Gurion University of the Negev, Beersheba, Israel
[5] Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, United Kingdom
E-mail: s.popescu@gmail.com and rohrlich@bgu.ac.il

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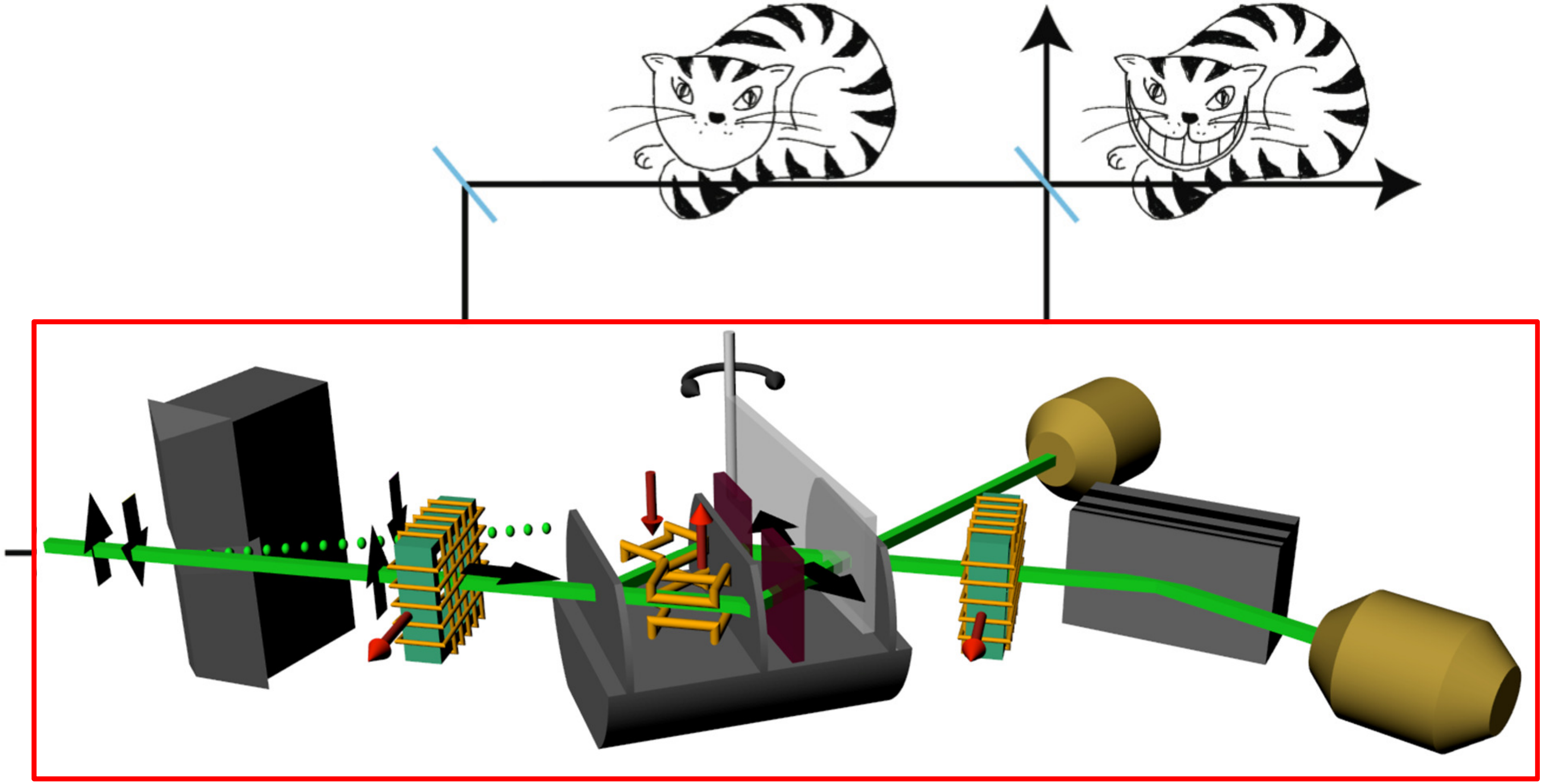
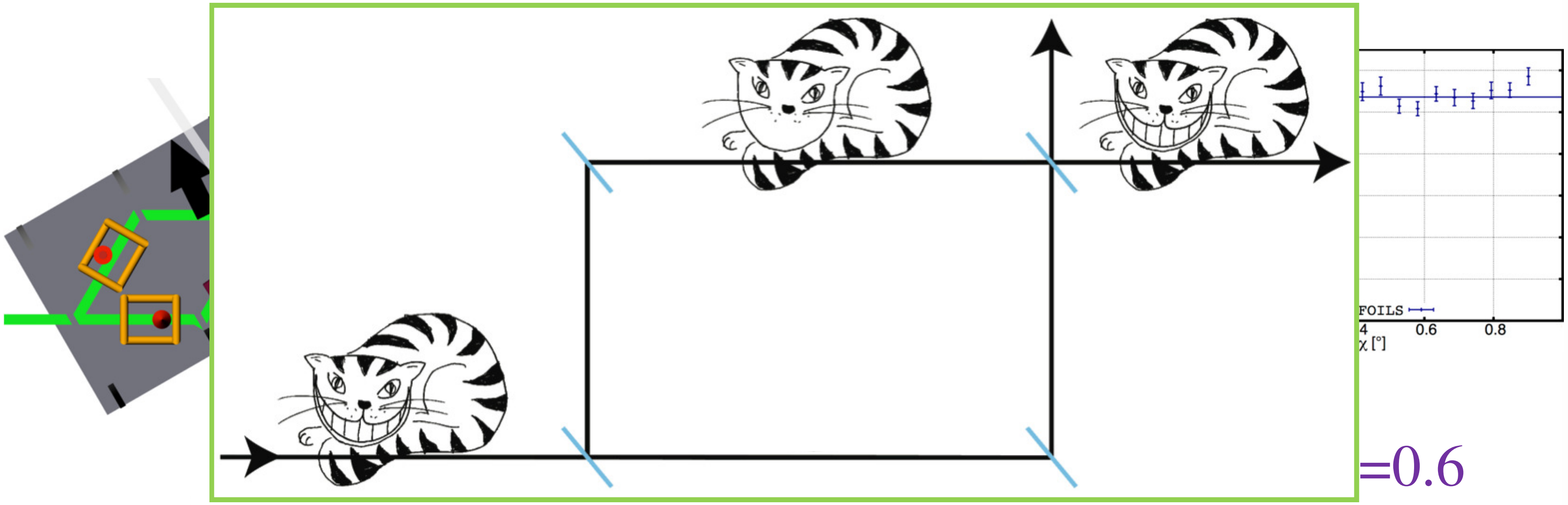
doi:10.1088/1367-2630/15/11/113015

**Abstract.** In this paper we present a quantum Cheshire Cat. In a pre- and post-selected experiment we find the Cat in one place, and its grin in another. The Cat is a photon, while the grin is its circular polarization.

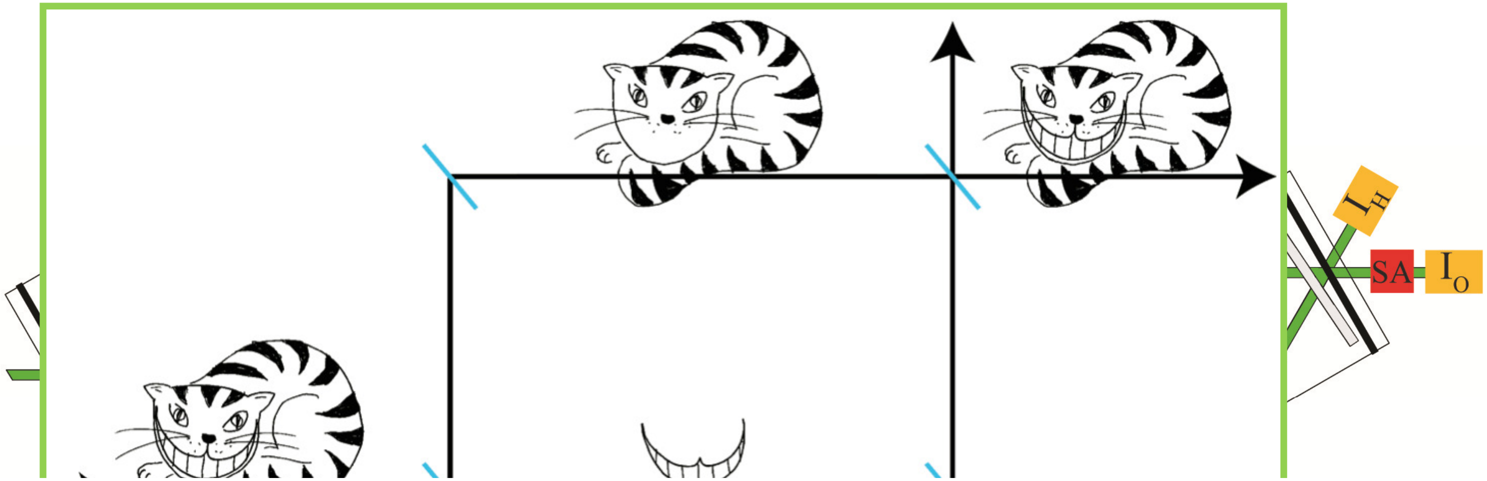
# Quantum Cheshire-cat in neutron interferometer



# Quantum Cheshire-cat: neutron(cat) in upper path



# Quantum Cheshire-cat: spin(smile) in lower path



# Weak measurement, weak value

VOLUME 60, NUMBER 14 PHYSICAL REVIEW LETTERS 4 APRIL 1988

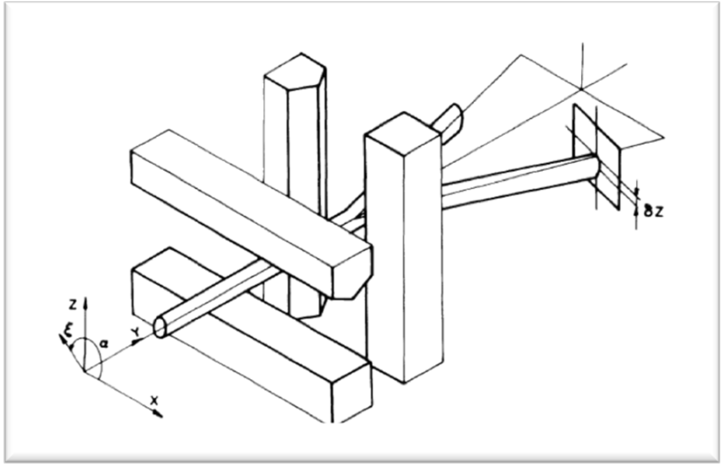
**How the Result of a Measurement of a Component of the Spin of a Spin-$\frac{1}{2}$ Particle Can Turn Out to be 100**

Yakir Aharonov, David Z. Albert, and Lev Vaidman

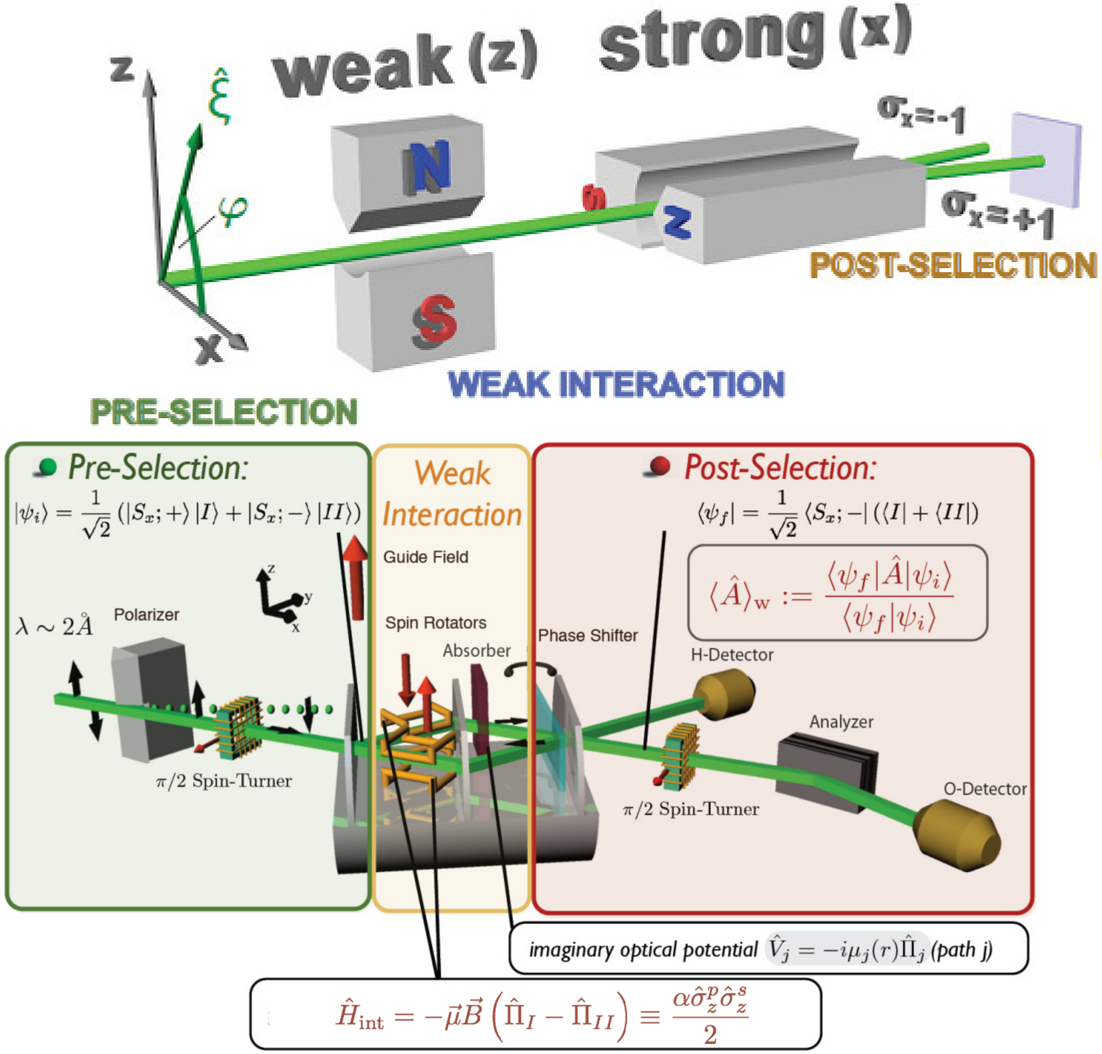
*Physics Department, University of South Carolina, Columbia, South Carolina 29208, and School of Physics and Astronomy, Tel-Aviv University, Ramat Aviv 69978, Israel*

(Received 30 June 1987)

We have found that the usual measuring procedure for preselected and postselected ensembles of quantum systems gives unusual results. Under some natural conditions of weakness of the measurement, its result consistently defines a new kind of value for a quantum variable, which we call the weak value. A description of the measurement of the weak value of a component of a spin for an ensemble of preselected and postselected spin-$\frac{1}{2}$ particles is presented.

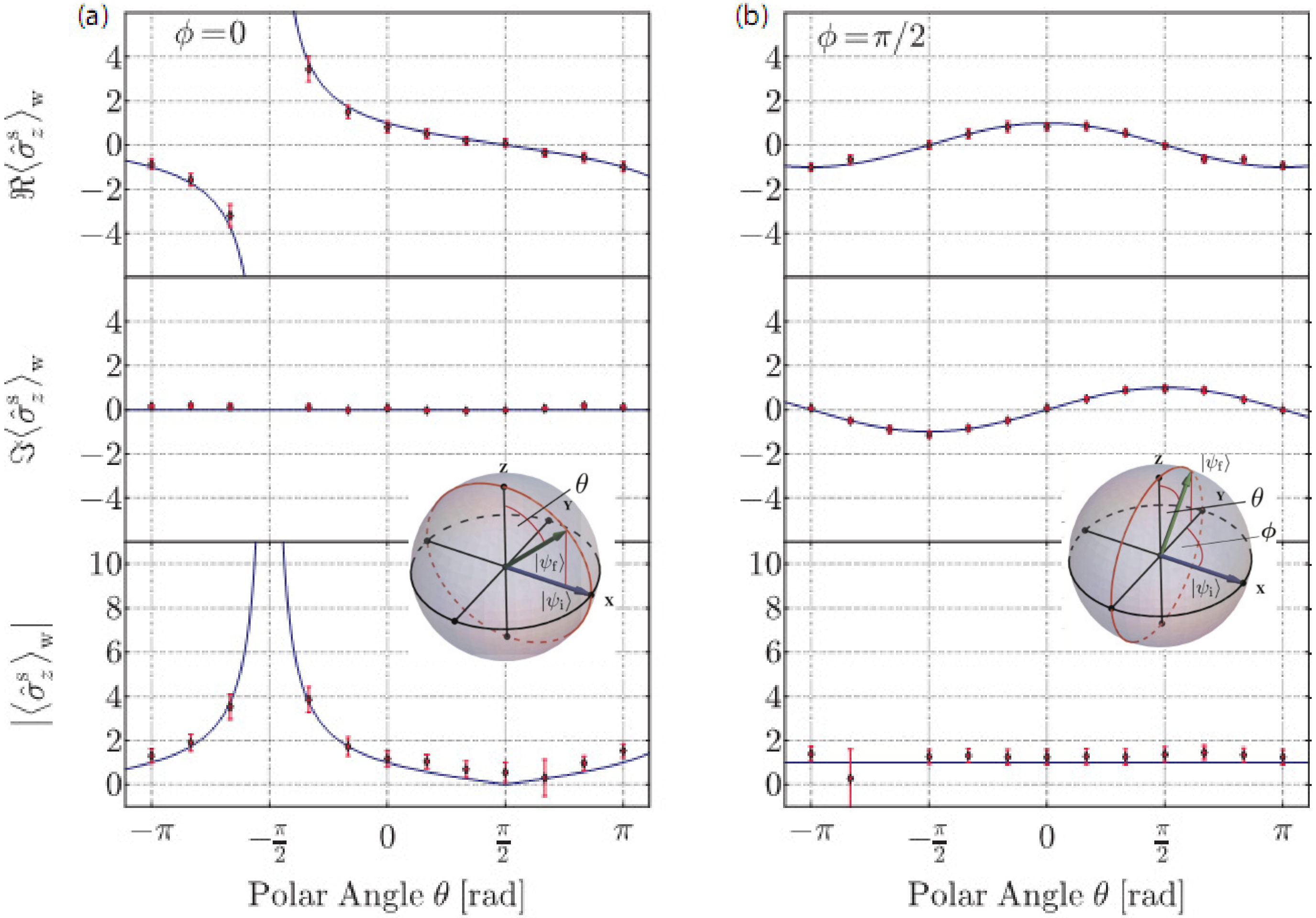


# Weak measurement of neutron's ½-spin




$$\left\langle \hat{\sigma}_z^s \right\rangle_w = \frac{\left\langle s_f \middle| \hat{\sigma}_z \middle| s_i \right\rangle}{\left\langle s_f \middle| s_i \right\rangle}$$

# Weak measurement of ½-spin: final results




S. Sponar et al. arXiv. Quant-ph/1404.2125

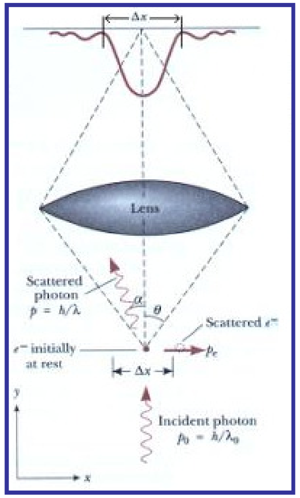
# Uncertainty relation: historical

- In 1927 Heisenberg postulated an uncertainty principle:

  γ-ray thought experiment

  $$\rightarrow \quad p_1 q_1 \approx h$$

  with $q_1$ (mean error) & $p_1$ (discontinuous change)




- Sei $q_1$ die Genauigkeit, mit der der Wert $q$ bekannt ist ($q_1$ ist etwa der mittlere Fehler von $q$), also hier die Wellenlänge des Lichtes, $p_1$ die Genauigkeit, mit der der Wert $p$ bestimmbar ist, also hier die unstetige Änderung von $p$ beim Comptoneffekt, so stehen nach elementaren Formeln des Comptoneffekts $p_1$ und $q_1$ in der Beziehung

$$p_1 q_1 \sim h. \qquad (1)$$

# Ozawa's Universally Valid Uncertainty Relation

PHYSICAL REVIEW A **67**, 042105 (2003)

**Universally valid reformulation of the Heisenberg uncertainty principle on noise and disturbance in measurement**

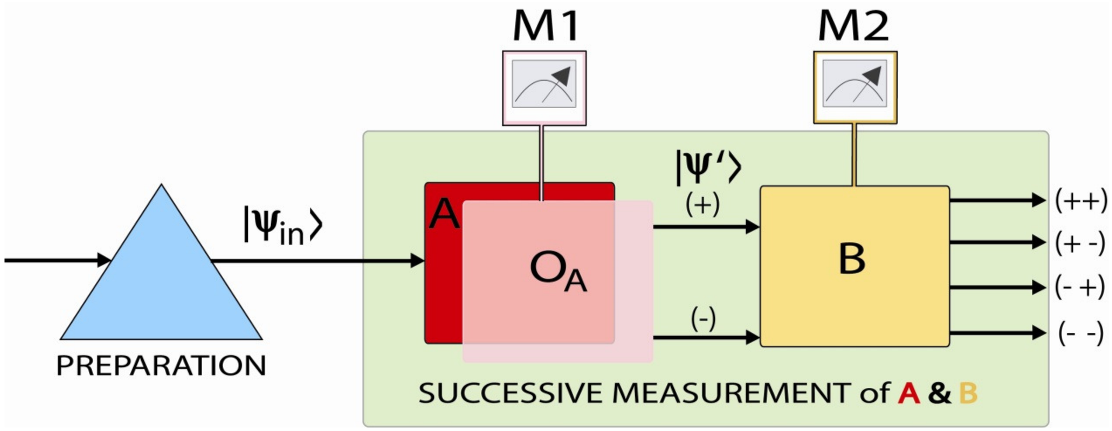
Masanao Ozawa
*Graduate School of Information Sciences, Tôhoku University, Aoba-ku, Sendai, 980-8579, Japan*
(Received 9 October 2002; published 11 April 2003)

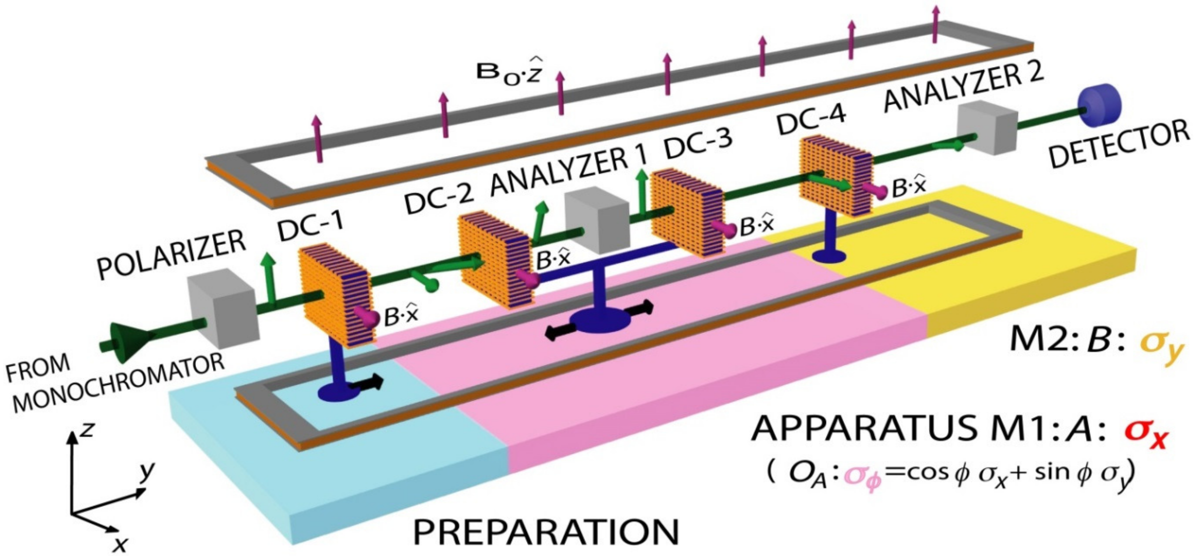
The Heisenberg uncertainty principle states that the product of the noise in a position measurement and the momentum disturbance caused by that measurement should be no less than the limit set by Planck's constant $\hbar/2$ as demonstrated by Heisenberg's thought experiment using a $\gamma$-ray microscope. Here it is shown that this common assumption is not universally true: a universally valid trade-off relation between the noise and the disturbance has an additional correlation term, which is redundant when the intervention brought by the

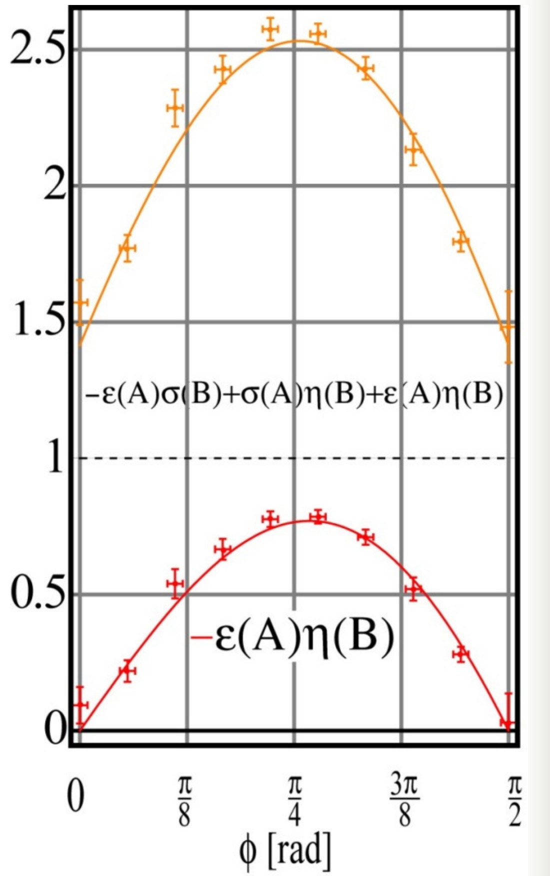
$$\epsilon(A)\eta(B) + \epsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle|$$

- rigorous theoretical treatments of quantum measurements:
  - ***first term:*** error of the first measuremt, disturbance on the second measurement
  - ***second and third terms:*** crosstalks between spreads of wavefunctions and error/disturbance

# Experimental test




J. Erhart et al.,
Nature Phys. **8**, 185-189 (2012)

# Publications by other groups

PRL 109, 100404 (2012) PHYSICAL REVIEW LETTERS week ending 7 SEPTEMBER 2012

### Violation of Heisenberg's Measurement-Disturbance Relationship by Weak Measurements

Lee A. Rozema, Ardavan Darabi, Dylan H. Mahler, Alex Hayat, Yasaman Soudagar, and Aephraim M. Steinberg

PRL 110, 220402 (2013) PHYSICAL REVIEW LETTERS week ending 31 MAY 2013

### Experimental Test of Universal Complementarity Relations

Morgan M. Weston, Michael J. W. Hall, Matthew S. Palsson, Howard M. Wiseman, and

*Centre for Quantum*

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ArXiv; 1304.2071

### How well can one jointly measure two incompatible observables on a given quantum state?

Cyril Branciard

*Centre for Engineered Quantum Systems and Scho*

*The University of Queensland, St Lucia,*

(Dated: April 9, 2013

Heisenberg's uncertainty principle is one of the main tenets of
fundamental importance for our understanding of quantum foun
interpretation: although Heisenberg's first argument was that the r
state necessarily disturbs another incompatible observable, stand
indeterminacy of the outcomes when either one or the other obse
precisely Heisenberg's intuition. Even if two incompatible obse
still approximate their joint measurement, at the price of introd
measurement of each of them. We present a new, tight relation
the error on one observable versus the error on the other. As
characterize the disturbance of an observable induced by the appr
derive a stronger error-disturbance relation for this scenario.

SCIENTIFIC REPORTS

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SUBJECT AREAS:
QUANTUM MECHANICS
QUANTUM METROLOGY
QUANTUM INFORMATION
QUANTUM OPTICS

Received 7 August 2012

Accepted 2 July 2013

Published 17 July 2013

### Experimental violation and reformulation of the Heisenberg's error-disturbance uncertainty relation

So-Young Baek[1]*, Fumihiro Kaneda[1], Masanao Ozawa[2] & Keiichi Edamatsu[1]

[1]Research Institute of Electrical Communication, Tohoku University, Sendai 980-8577, Japan, [2]Graduate School of Information Science, Nagoya University, Nagoya 464-8601, Japan.

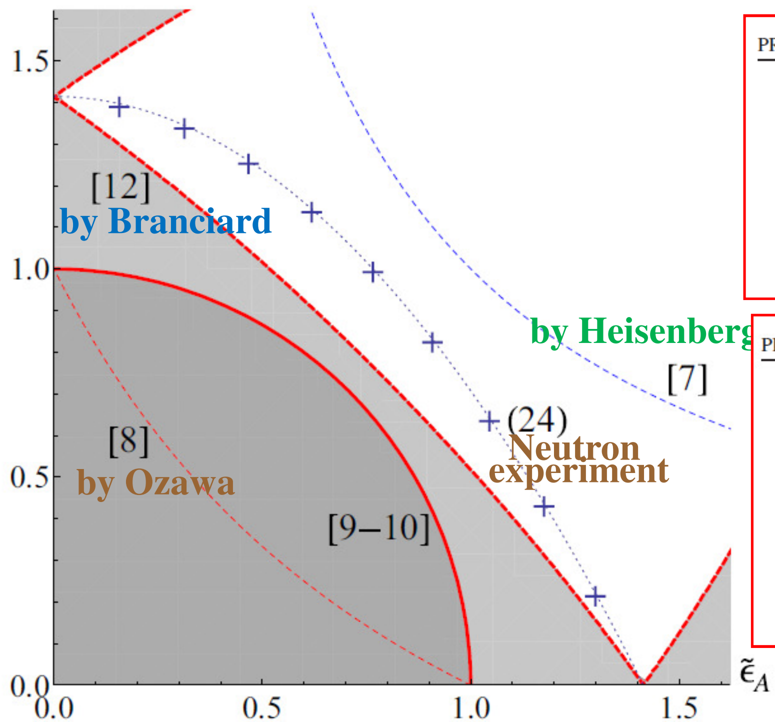
Correspondence and requests for materials

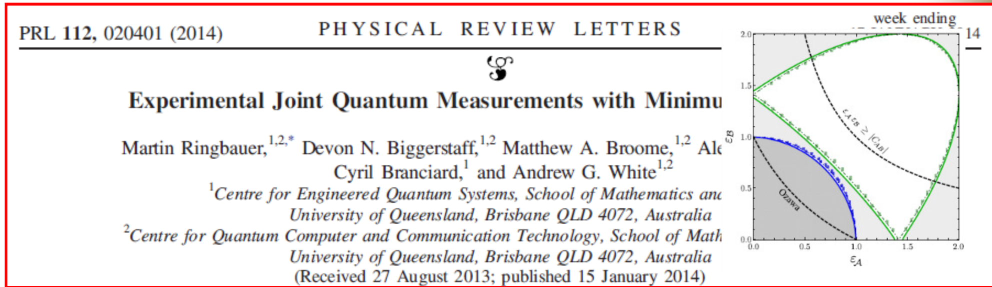
The uncertainty principle formulated by Heisenberg in 1927 describes a trade-off between the error of a measurement of one observable and the disturbance caused on another complementary observable such that their product should be no less than the limit set by Planck's constant. However, Ozawa in 1988 showed a model of position measurement that breaks Heisenberg's relation and in 2003 revealed an alternative relation for error and disturbance to be proven universally valid. Here, we report an experimental test of Ozawa's relation for a single-photon polarization qubit, exploiting a more general class of quantum measurements than the class of projective measurements. The test is carried out by linear optical devices and realizes an indirect measurement model that breaks Heisenberg's relation throughout the range of our experimental parameter and yet validates Ozawa's relation.

# Tight relation derived by Branciard

$$\left[2\tilde{\varepsilon}(A)\tilde{\eta}(B)\sqrt{1-C^2}+\tilde{\varepsilon}(A)^2+\tilde{\eta}(B)^2\right]^{1/2}\geq C,$$

$$where\ \tilde{\varepsilon}=\varepsilon\sqrt{1-\varepsilon^2/4},\ \tilde{\eta}=\eta\sqrt{1-\eta^2/4},\ C\equiv\left|\langle\psi|[A,B]|\psi\rangle\right|/2$$

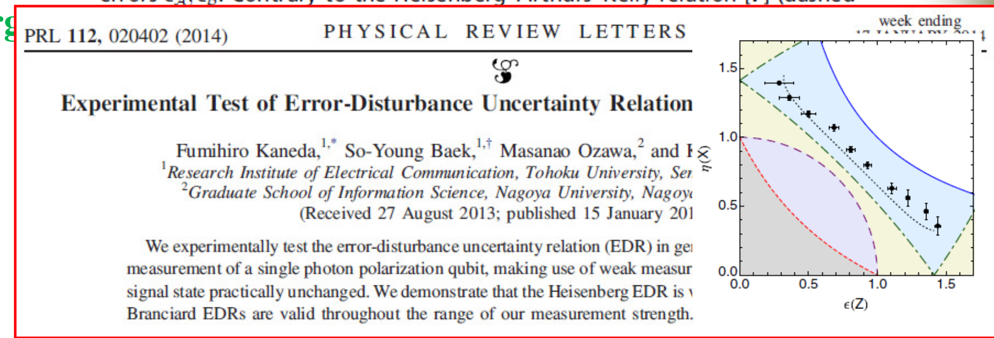


PRL 112, 020401 (2014) PHYSICAL REVIEW LETTERS week ending

**Experimental Joint Quantum Measurements with Minimu**

Martin Ringbauer,[1,2,*] Devon N. Biggerstaff,[1,2] Matthew A. Broome,[1,2] Al
Cyril Branciard,[1] and Andrew G. White[1,2]
[1]*Centre for Engineered Quantum Systems, School of Mathematics an*
*University of Queensland, Brisbane QLD 4072, Australia*
[2]*Centre for Quantum Computer and Communication Technology, School of Math*
*University of Queensland, Brisbane QLD 4072, Australia*
(Received 27 August 2013; published 15 January 2014)



PRL 112, 020402 (2014) PHYSICAL REVIEW LETTERS week ending

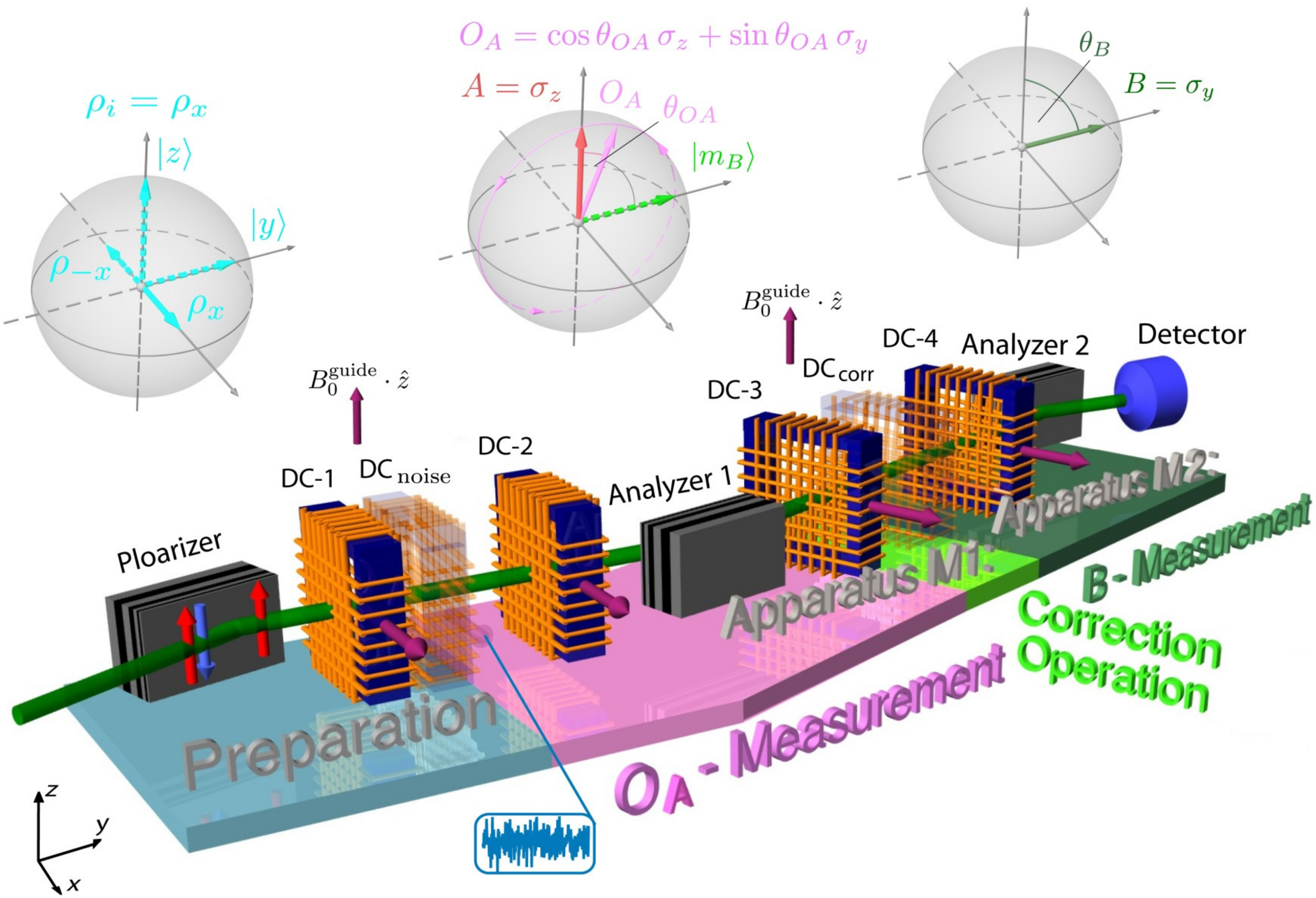
**Experimental Test of Error-Disturbance Uncertainty Relation**

Fumihiro Kaneda,[1,*] So-Young Baek,[1,†] Masanao Ozawa,[2] and I
[1]*Research Institute of Electrical Communication, Tohoku University, Se*
[2]*Graduate School of Information Science, Nagoya University, Nagoy*
(Received 27 August 2013; published 15 January 201

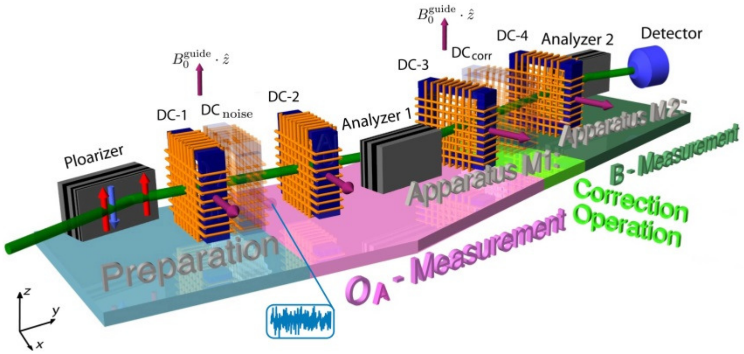
We experimentally test the error-disturbance uncertainty relation (EDR) in ge
measurement of a single photon polarization qubit, making use of weak measur
signal state practically unchanged. We demonstrate that the Heisenberg EDR is
Branciard EDRs are valid throughout the range of our measurement strength.

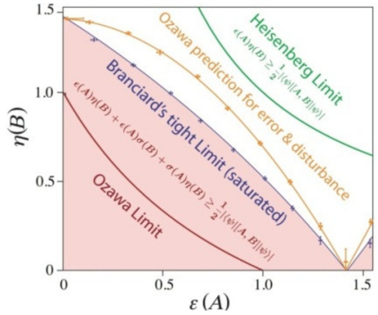
C. Branciard, Proc. Natl. Acad. Sci. U.S.A. **110**, 6742 (2013).

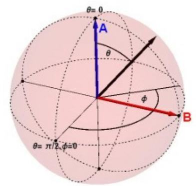
# Tight relation: experimental setup

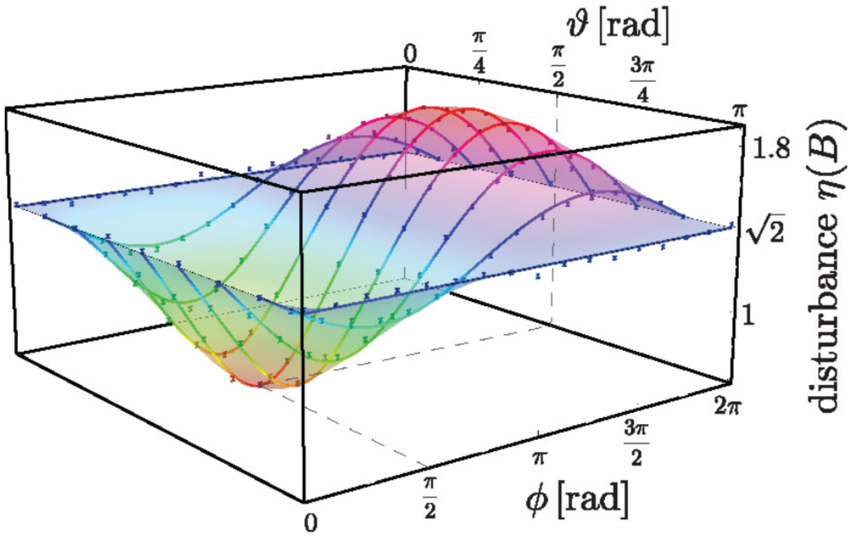


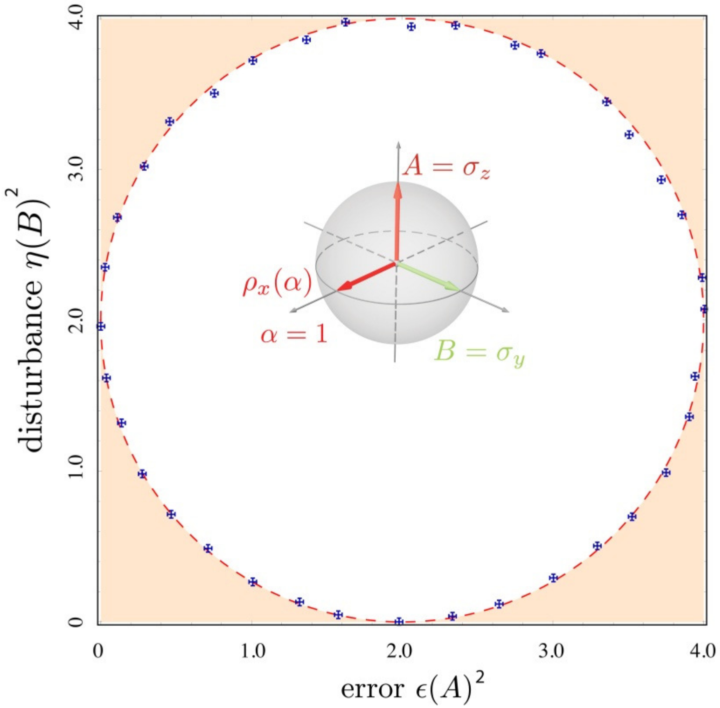
# Tight relation: error-corrections



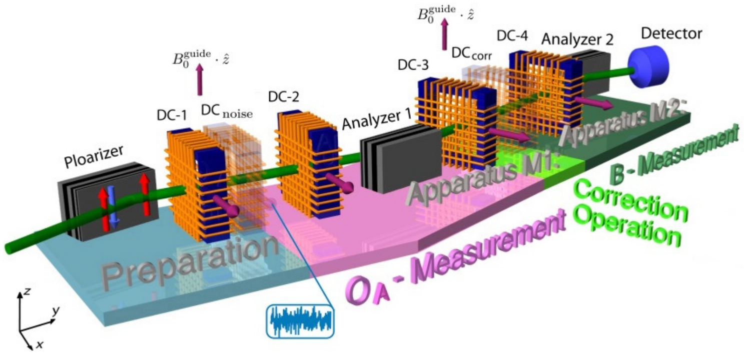



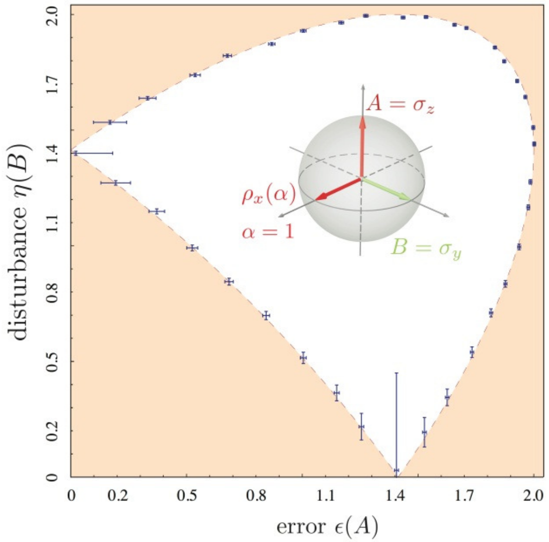



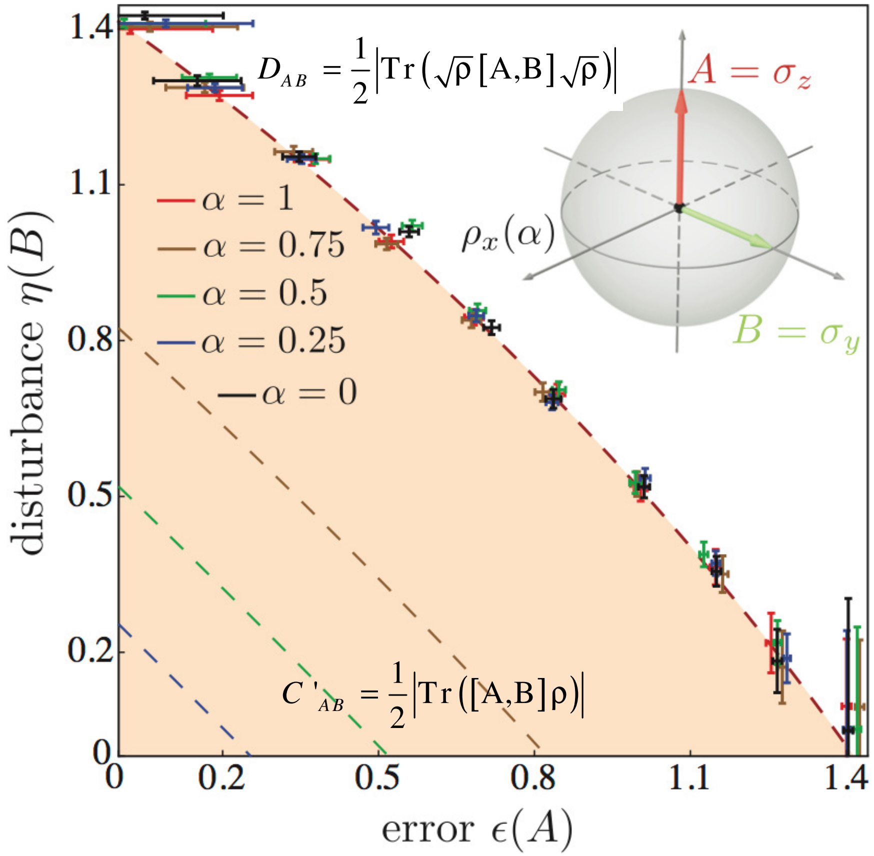



# Tight relation: from a pure state to mixed states

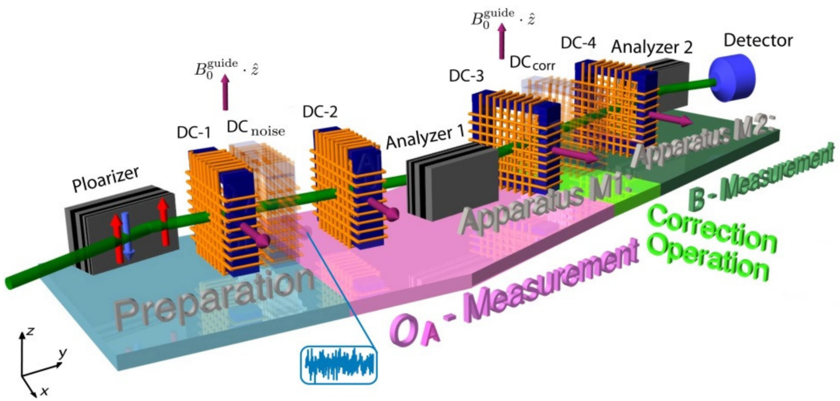


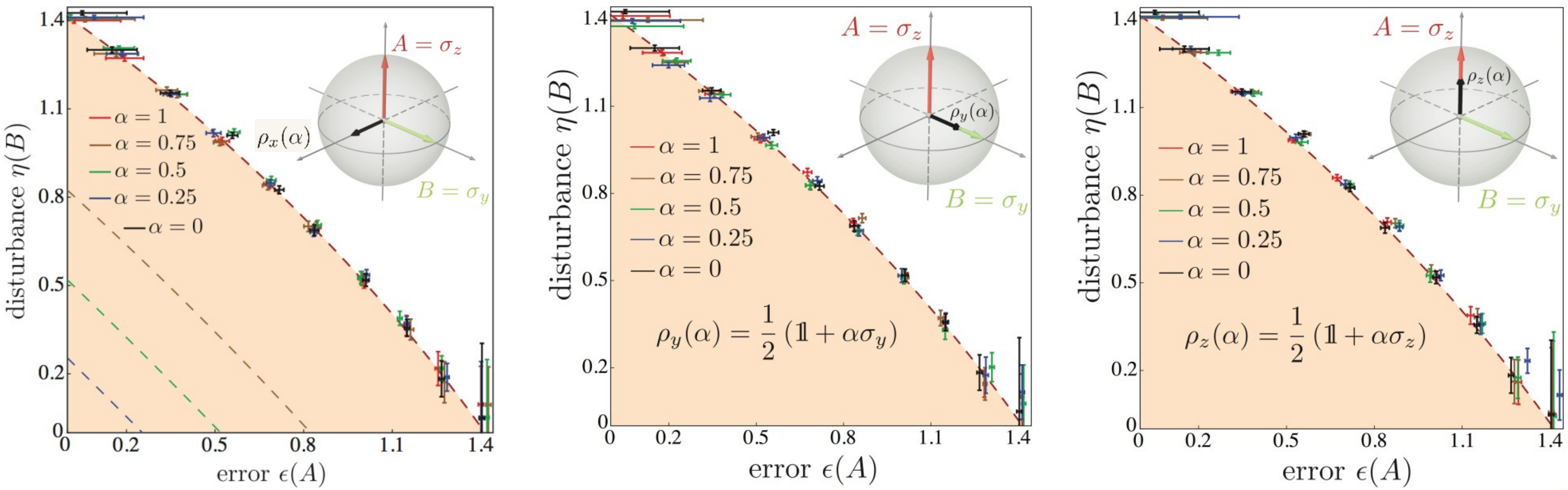




Remark: M. Ozawa, arXiv:1404.3388

# Tight relation: all mixtures



# Entropic uncertain-relation (UR)

## UR for states

❖ Robertson:

$$\Delta A\, \Delta B \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle|$$

❖ Deutsch:

$$H(\mathcal{A}) + H(\mathfrak{B}) \geq -2\log(c)$$

$$c := \max_{j,k}|\langle a_j|b_k\rangle|$$

## UR for measurements

❖ Ozawa: *arXiv:* 1404.3388v1, (2014)

$$\varepsilon(A)^2\Delta B^2 + \eta(B)^2\Delta A^2 + 2\varepsilon(A)\eta(B)\sqrt{\Delta A^2\Delta B^2 - {D_{AB}}^2} \geq {D_{AB}}^2$$

$$D_{AB} := \frac{1}{2}\mathrm{Tr}(|\sqrt{\rho}[A,B]\sqrt{\rho}|)$$

❖ Buscemi, Hall: PRL **112**, 050401 (2014)

$$N(M,A) + D(M,B) \geq -\log(c)$$

$$N(M,A) := H(\mathcal{A}|\mathcal{M}) \;\&\; D(M,B) := H(\mathfrak{B}|\mathcal{M})$$

# Information-theoretic Entropy

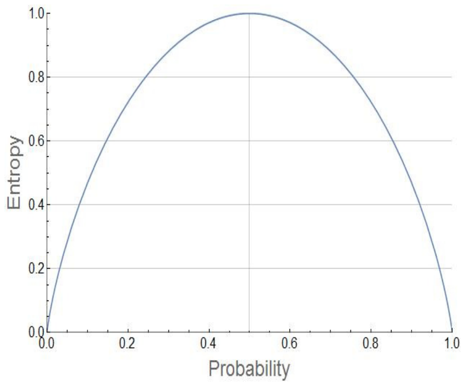
Shannon Entropy *H*:

where $A|a\rangle = \mathrm{a}|a\rangle$ for the <u>observable A.</u>

$$H(\mathcal{A}, |\psi\rangle ) := -\sum_a p(a)\log\left(p(a)\right)$$

$$p(a) = |\langle \mathrm{a}|\psi\rangle|^2$$

Coin toss: Probability for heads or tails

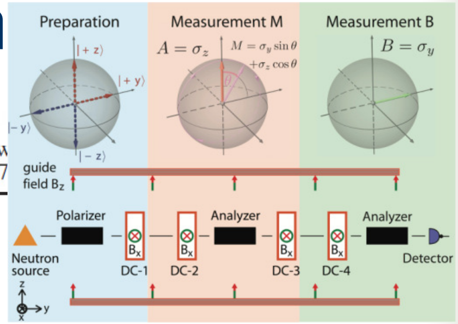



p(heads) = x p(tails) = 1-x

(Binary) Shannon entropy

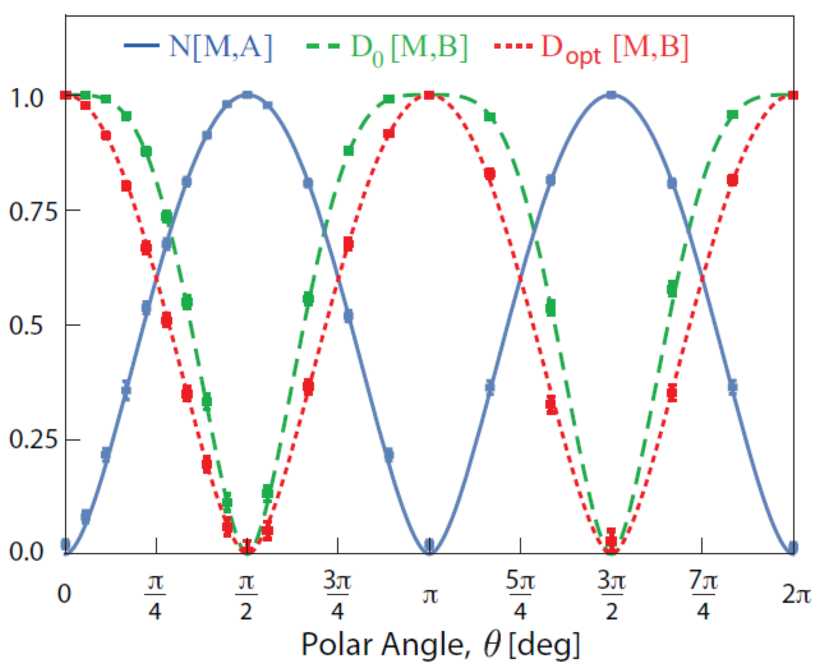
$$H(\mathrm{X}) = -\mathrm{x}\log(\mathrm{x}) - (1-\mathrm{x})\log(1-\mathrm{x})$$

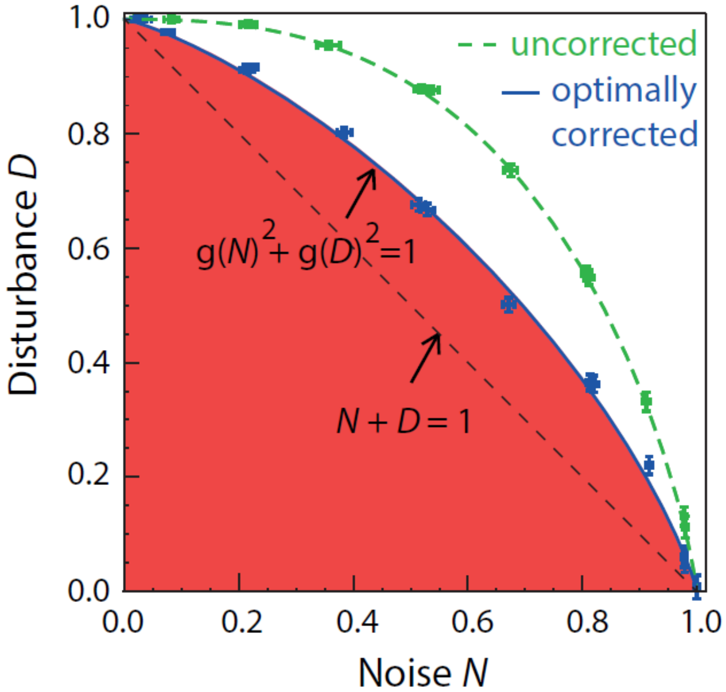
# Results for entropic noise-dist. relation




PRL 115, 030401 (2015) PHYSICAL REVIEW LETTERS

**Experimental Test of Entropic Noise-Disturbance Uncertainty Relations for Spin-1/2 Measurements**




**Entropic noise-dist. uncertainty relation has π-periodicity !!!**

**Tight relation is attained.**

# Concluding remarks

**Neutron optical method is an effective strategy for studies of foundation of quantum mechanics.**

**- Quantum dyanamics: quantum Cheshire-cat and full weak-value determination are demonstrated.**

**- Error-disturbance uncertainty relation: tight relation for pure/mixed input-states are shown.**

**- Entropic noise-disturbance uncertainty relation: tightness of the relation is confirmed.**

*Neutron Quantum Optics generation*

# Fin !

# The neutron

**Particle** | Feels four-forces | **Wave**

$m = 1.674928\,(1) \times 10^{-27}$ kg

$s = \frac{1}{2} h$

$\mu = -9.6491783(18) \times 10^{-27}$ J/T

$\tau = 887(2)$ s

$R = 0.7$ fm

$\alpha = 12.0\,(2.5) \times 10^{-4}$ fm$^3$

u - d - d - quark structure

**CONNECTION**

de Broglie

$$\lambda_B = \frac{h}{m.v}$$

Schrödinger

$$H\psi(\vec{r},t) = i\hbar \frac{\delta\psi(\vec{r},t)}{\delta t}$$

&

boundary conditions

$$\lambda_c = \frac{h}{m.c} = 1.319695\,(20) \times 10^{-15} \text{ m}$$

For thermal neutrons
= 2 Å, 2000 m/s, 20meV

$$\lambda_B = \frac{h}{m.v} = 1.8 \times 10^{-10} \text{ m}$$

$$\Delta_c = \frac{1}{2\delta k} \cong 10^{-8} \text{ m}$$

$$\Delta_p = v.\Delta t \cong 10^{-2} \text{ m}$$

$$\Delta_d = v.\tau = 1.942(5) \times 10^6 \text{ m}$$

$$0 \le \chi \le 2\pi\ (4\pi)$$

m ... mass, s ... spin, μ ... magnetic moment, τ ... β-decay lifetime, R ... (magnetic) confinement radius, α ... electric polarizability; all other measured quantities like electric charge, magnetic monopole and electric dipole moment are compatible with zero

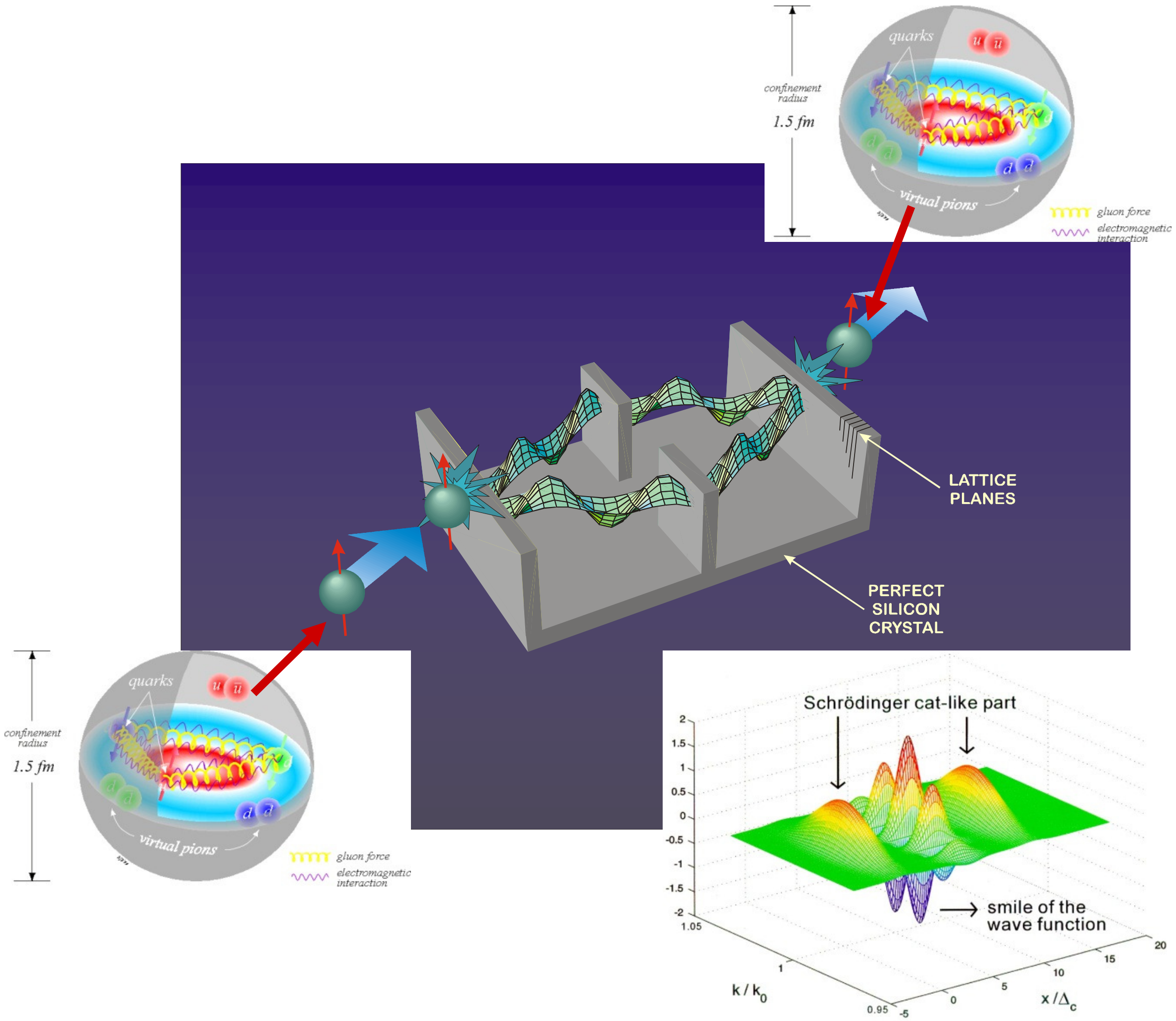
$-\mu B$ ________________

$\mu B$ ________________

two level system

$\lambda_c$ ... Compton wavelength, $\lambda_B$ ... deBroglie wavelength, $\Delta_c$ ... coherence length, $\Delta_p$ ... packet length, $\Delta_d$ ... decay length, δk.... momentum width, Δt ... chopper opening time, v ... group velocity, χ ... ... phase.

# Neutron interferometer




# Two-particle vs. two-space entanglement

## 2-Particle Bell-State

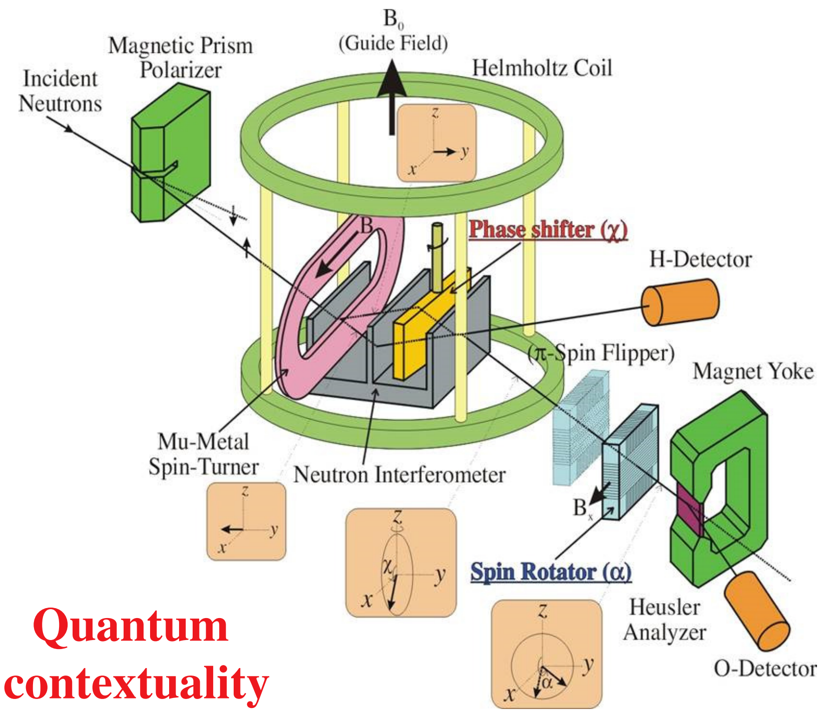
$$|\Psi\rangle = \frac{1}{\sqrt{2}}\{|\uparrow\rangle_{\mathrm{I}}\otimes|\downarrow\rangle_{\mathrm{II}} + |\downarrow\rangle_{\mathrm{I}}\otimes|\uparrow\rangle_{\mathrm{II}}\}$$

I, II represent 2-Particles

## 2-Space Bell-State

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\{|\uparrow\rangle_{\mathrm{s}}\otimes|\mathrm{I}\rangle_{\mathrm{p}} + |\downarrow\rangle_{\mathrm{s}}\otimes|\mathrm{II}\rangle_{\mathrm{p}}\}$$

s, p represent 2-Spaces, e.g., spin & path




**Quantum contextuality**

**Violation of Bell-like inequality**

$$S' \equiv E'(\alpha_1,\chi_1) + E'(\alpha_1,\chi_2) - E'(\alpha_2,\chi_1) + E'(\alpha_2,\chi_2)$$

$$= \mathbf{2.051} \pm \mathbf{0.019} \quad > \mathbf{2}$$

**Hasegawa et al., Nature2003, NJP2011**

**Kochen-Specker-like contradiction 1**

$$E_x \cdot E_y = 0.407 \overset{63\%}{\longleftrightarrow} E' \equiv \left\langle \hat{X}_1 \hat{Y}_2 \cdot \hat{Y}_1 \hat{X}_2 \right\rangle = -0.861$$

**Hasegawa et al., PRL2006/2009**

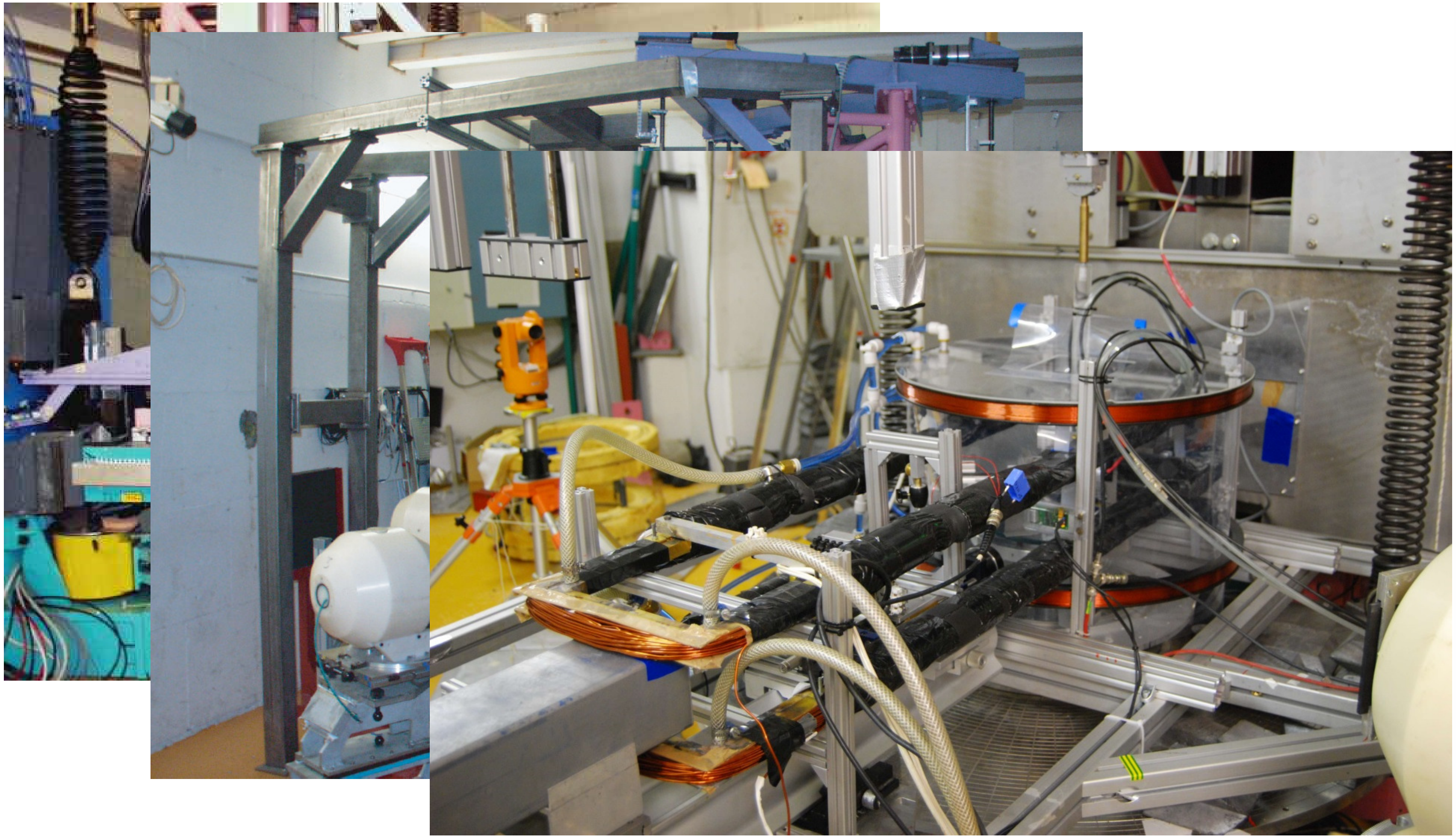
**Tri-partite entanglement (GHZ-state)**

$$\left|\Psi_{\mathrm{Neutron}}\right\rangle = \{|\Psi_{\mathrm{I}}\rangle \otimes |\uparrow\rangle \otimes |\Psi(E_0)\rangle + \left(e^{i\chi}|\Psi_{\mathrm{II}}\rangle\right) \otimes \left(e^{i\alpha}|\downarrow\rangle\right) \otimes \left(e^{i\gamma}|\Psi(E_0 + \hbar\omega_r)\rangle\right)\}$$

$$M_{Measured} = 2.558 \pm 0.004 \quad > 2$$

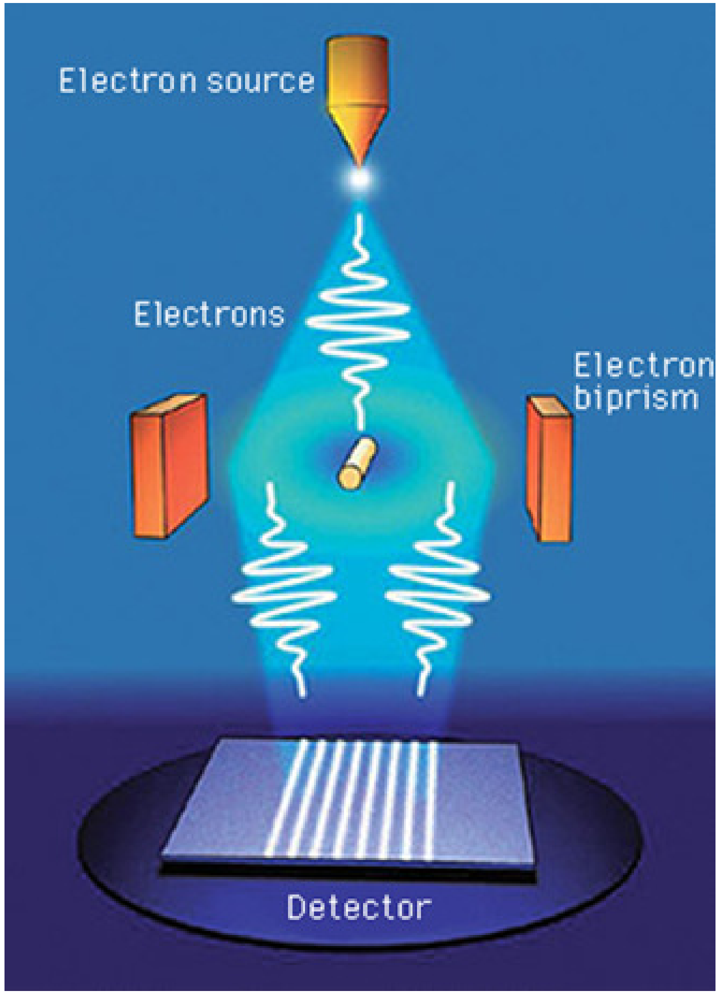
**Hasegawa et al., PRA2010**

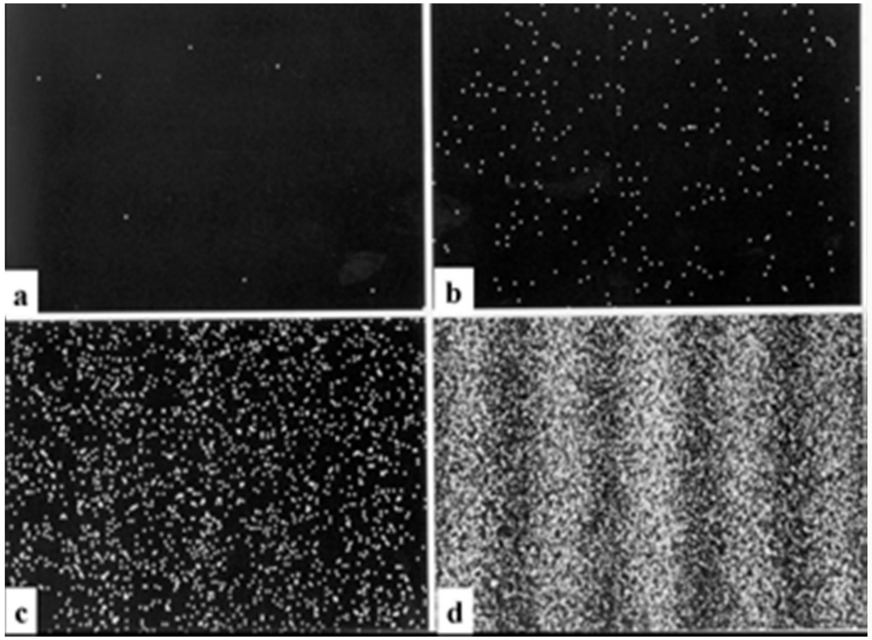
# Neutron interferometer beamline S18, ILL



**@ Institut Laue Langevin (ILL)**

# Wave-partcle duality with electrons



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# Particle/Wave in quantum mechanics

## *Particle and wave properties*

$$p = mv = h/\lambda$$

**(L. De Broglie)**

## *Schroedinger equation*

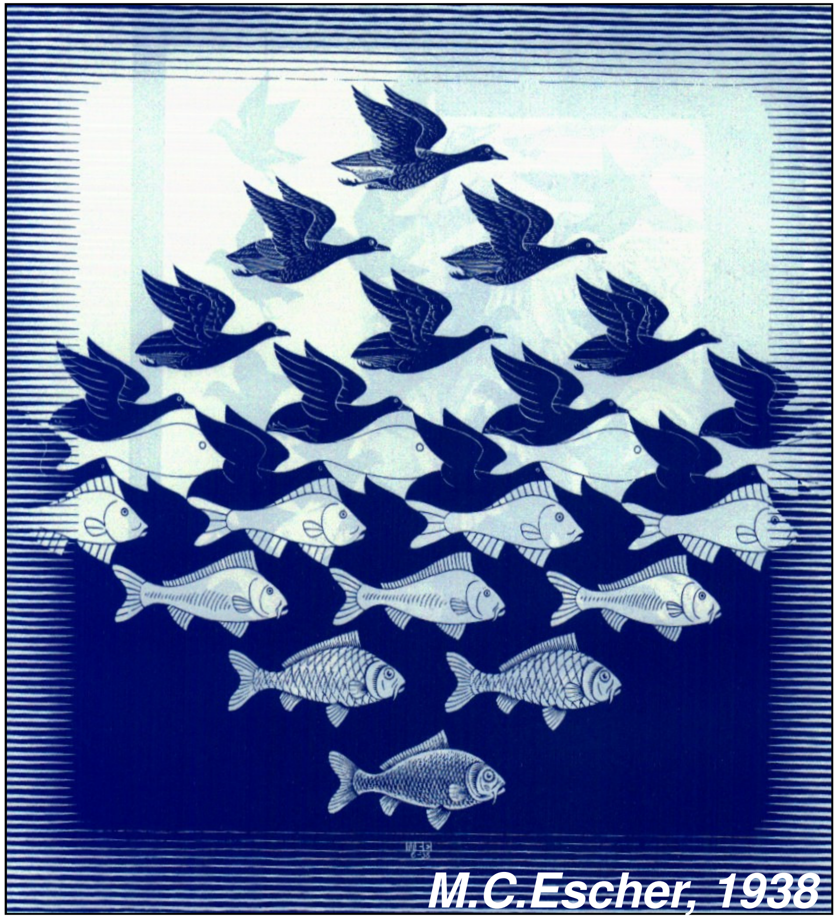
$$i\hbar \frac{\partial \Psi(\vec{r},t)}{\partial t} = H\Psi(\vec{r},t)$$

**(E. Schrödinger)**

## *Uncertainity*

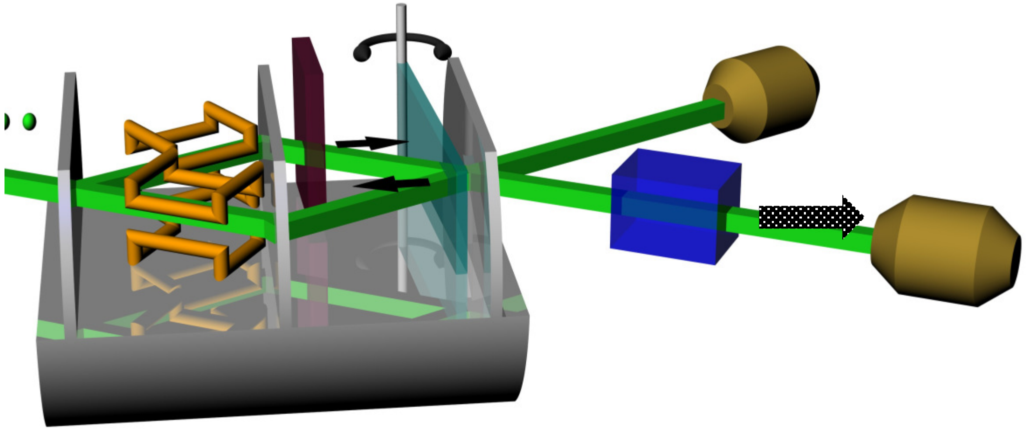
$$\Delta x\, \Delta p \geq h/4$$

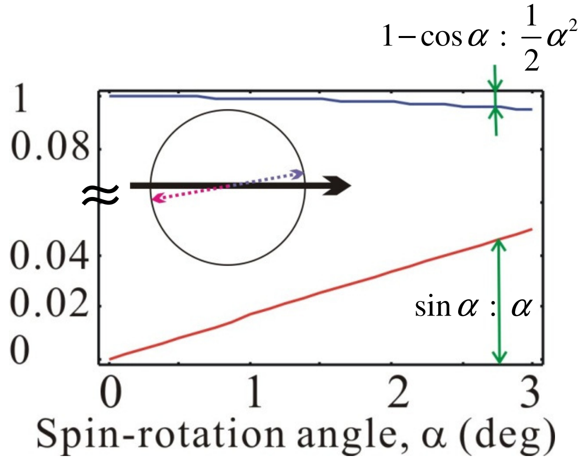
**(W. Heisenberg)**

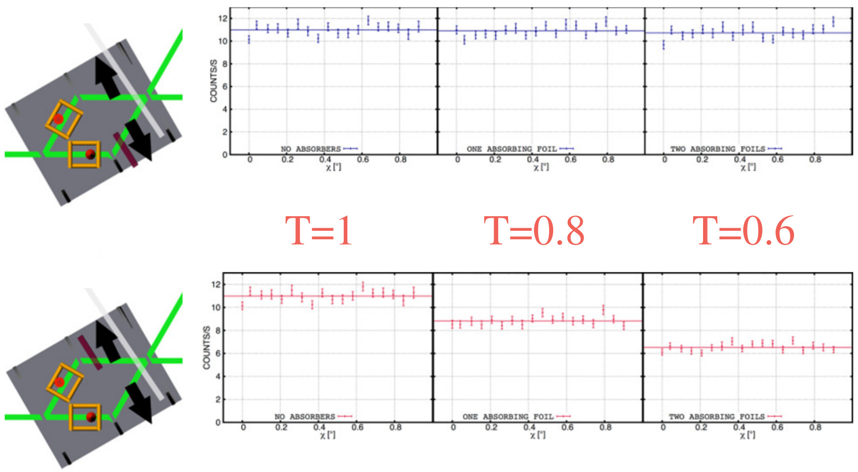


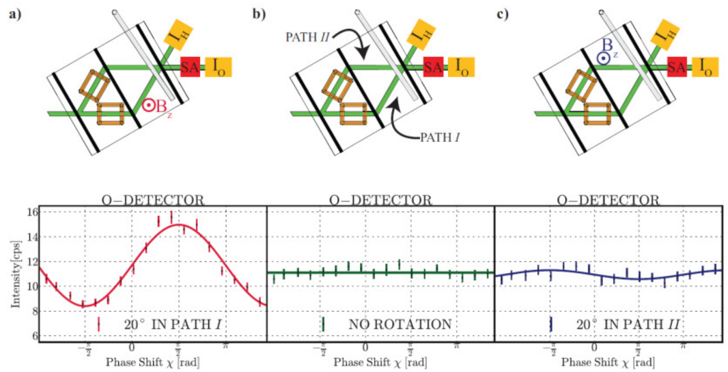
*M.C.Escher, 1938*

# Another view of quantum Cheshire-cat: effectiveness



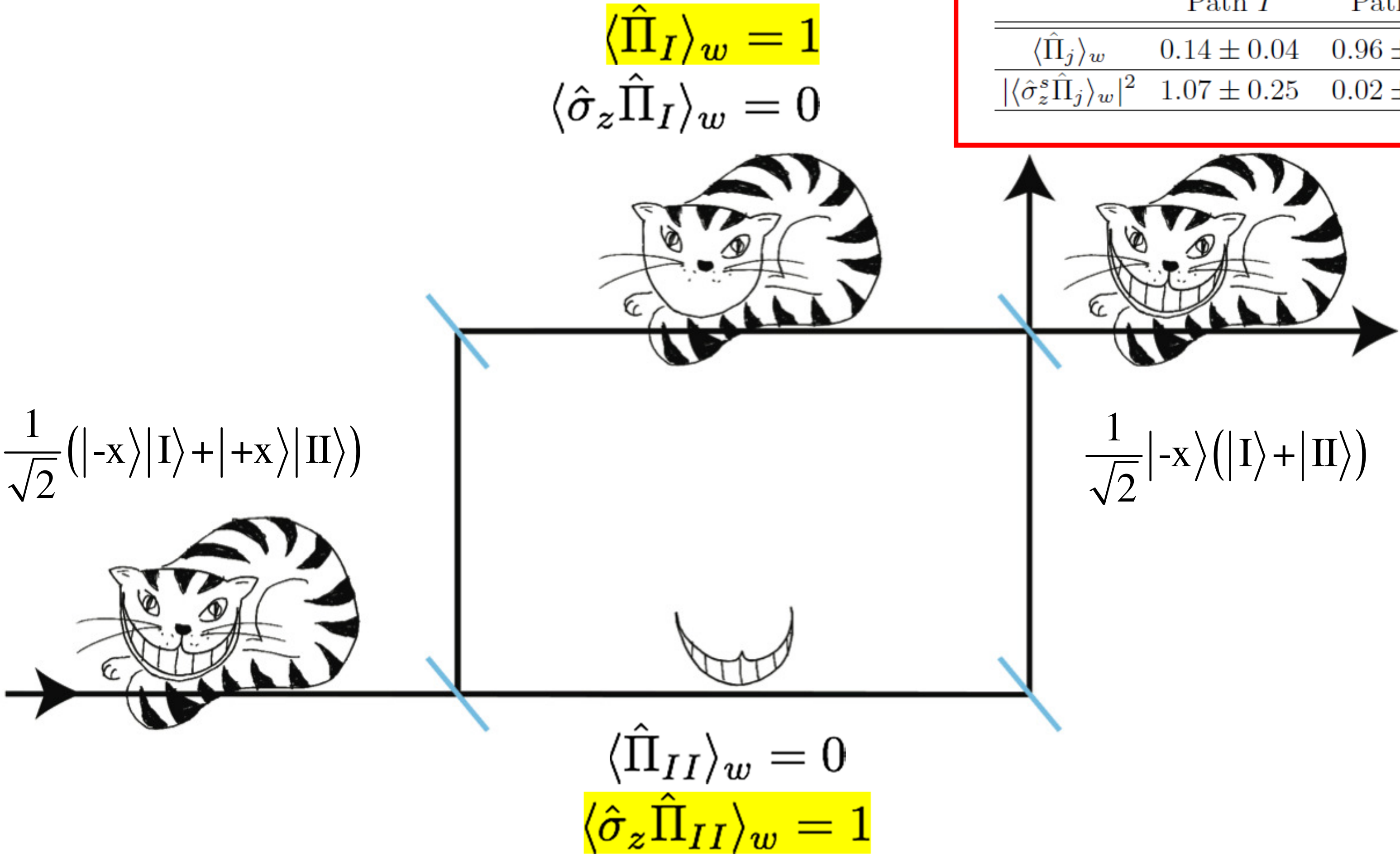


# Quantum Cheshire-cat: final results

| | Path $I$ | Path $II$ |
|---|---|---|
| $\langle \hat{\Pi}_j \rangle_w$ | $0.14 \pm 0.04$ | $0.96 \pm 0.06$ |
| $\lvert \langle \hat{\sigma}_z^s \hat{\Pi}_j \rangle_w \rvert^2$ | $1.07 \pm 0.25$ | $0.02 \pm 0.24$ |




T. Denkmayr et al. Nature Comm. 5:4492 (2014).

# “ex post facto” attack

PRL **111**, 160405 (2013) PHYSICAL REVIEW LETTERS week ending 18 OCTOBER 2013

- Error is defined as the root-mean-square (rms):

$$\varepsilon(A) \equiv \langle \psi \otimes \xi | [U^\dagger (I \otimes M) U - A \otimes I]^2 | \psi \otimes \xi \rangle^{1/2}$$

- Disturbance is defined in the same manner:

$$\eta(B) \equiv \langle \psi \otimes \xi | [U^\dagger (B \otimes I) U - B \otimes I]^2 | \psi \otimes \xi \rangle^{1/2}$$

**Definition:** $(\mathcal{K}, \xi, U, M)$ is a measuring process

$\mathcal{K}$: a Hilbert space,
$\xi$: a unit vector in $\mathcal{K}$,
$U$: a unitary operator on $\mathcal{H} \otimes \mathcal{K}$,
$M$: a self-adjoint operator on $\mathcal{K}$.

M. Ozawa,
Ann. Phys **311**, 350 (2004).

# "ex post facto" attack: case by Busch et al

In contrast, Fig. 2 shows the scenario discussed by Heisenberg. The middle row shows an approximate position measurement $Q'$ followed by a momentum measurement. How should we define the momentum disturbance and position error in this setup? The error of the approximate position measurement $Q'$ clearly refers to the comparison with an ideal measurement $Q$ as shown in the first row. For the momentum disturbance we can say the same: We have remarked that the momenta before and after the microscope interaction do not commute, so the difference makes no sense in the individual case. However, we can compare the *distributions* of the momenta measured after the position measurement (we call this effective measurement $P'$) with the *distribution* an ideal momentum

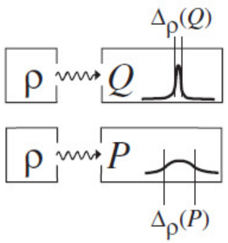



FIG. 1. Scenario of preparation uncertainty. $\Delta_\rho$ is the root of the variance of the distribution obtained for the indicated observable in the state $\rho$. In this pair of experiments no particle is subject to both a position and a momentum measurement.

merit for the device, a promise which might be advertised by the manufacturer, and which could be verified by a testing lab. $\Delta(Q, Q') = 0$ will mean that the "approximate" device $Q'$ is completely equivalent to the ideal $Q$; i.e., for every input state $\rho$ the output distributions will be

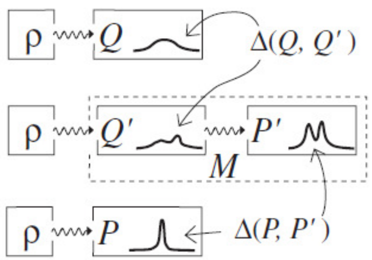



FIG. 2. Scenario of measurement uncertainty for successive measurements, as discussed by Heisenberg (middle row). An approximate position measurement $Q'$ is followed by an ideal momentum measurement, effectively given a measurement $P'$ on the initial state. The accuracy $\Delta(Q, Q')$ quantifies the difference between the output distributions of $Q'$ and an ideal position measurement $Q$ (first row). Similarly, the momentum disturbance $\Delta(P, P')$ quantifies the difference between the distributions obtained by $P'$ and by an ideal momentum measurement $P$ (last row). The definitions for these $\Delta$ quantities (see text) can be applied, more generally, to an arbitrary joint measurement $M$ (dashed box). This can be any device producing, in every shot, a $q$ value and a $p$ value. $Q'$ and $P'$ are then defined as the marginals of $M$, obtained by ignoring the other output.

160405-2

P. Busch et al. Phys. Rev. Lett. 106 (2013) 110406.

# "ex post facto" attack: case by Korzekwa et al etc.

s inevitable only between state-independent measures of

PACS number(s): 03.65.Ta, 03.67.−a

*Definition 1.* Operational disturbance: Consider a nonselective measurement of observable $A$ on a system in state $\rho$ that results in final state $\rho'$. We say the measurement of $A$, given $\rho$, is operationally disturbing to a subsequent measurement of $B$ iff the statistics of $B$ differ for $\rho$ and $\rho'$.

Moreover, any measure of disturbance should assign the value 0 to operationally nondisturbing measurements, which is the central *operational requirement* (OR) of this work. This is clearly an uncontroversial demand, however the reason we spell it out explicitly here is precisely that there are recent prominent examples [3,17–19] in the literature that fail to adhere t… satisfyin…

…a qubit …jective …nent of …ared in …r, then …, as the …tate.

phases between eigenstates of $B$ (the off-diagonal terms) should not be treated as disturbance to the measurement of $B$. In other words, disturbance of the measurement of $B$ occurs if and only if diagonal elements of $\rho$ in the basis of eigenstates of $B$ change. This is the essence of the OR, which is operationally motivated by the fact that only the change in the measurement statistics can be detected by the measurement (otherwise

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Physics Letters A 320 (2004) 261–270
PHYSICS LETTERS A
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Noise and disturbance in quantum measurement

Paul Busch [a], Teiko Heinonen [b,*], Pekka Lahti [c]

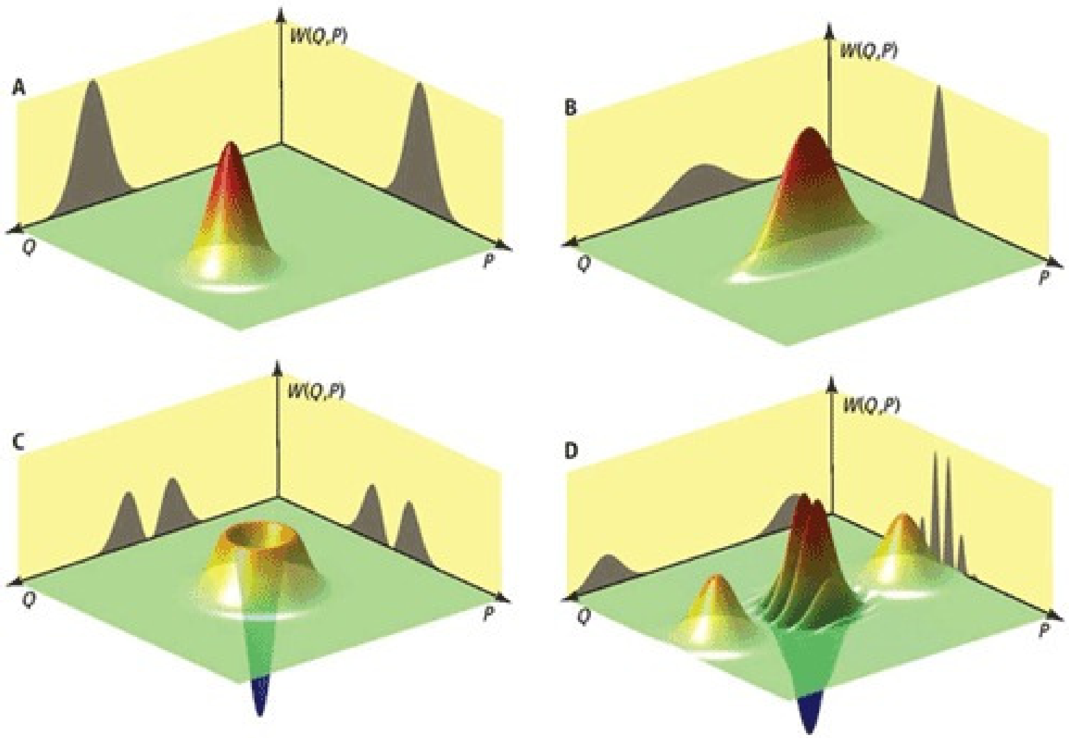
and the two observables involved. Second, whenever the noise is 'small', this should mean that the measurement is 'good'. We take this to mean that vanishing noise (in a given state $\psi$) should indicate that the probability distributions $E_\psi$ and $E_\psi^{\mathcal{M}}$ of $E$ and $E^{\mathcal{M}}$ are the same in that state. Finally, if the measurement is a good one, meaning that $E_\psi = E_\psi^{\mathcal{M}}$ for all states $\psi$, then this should be indicated by a vanishing noise measure for all $\psi$.

K. Korzekwa et al. Phys. Rev. A 89 (2014) 052108.

# “ex post facto” attack: functional vs operational

Wigner function, W(Q,P)

➔ quasi-probability distribution




Quantum state

➔ Quantum process

<given by density operators>

Measurement output

➔ functional

<classical phenomena>

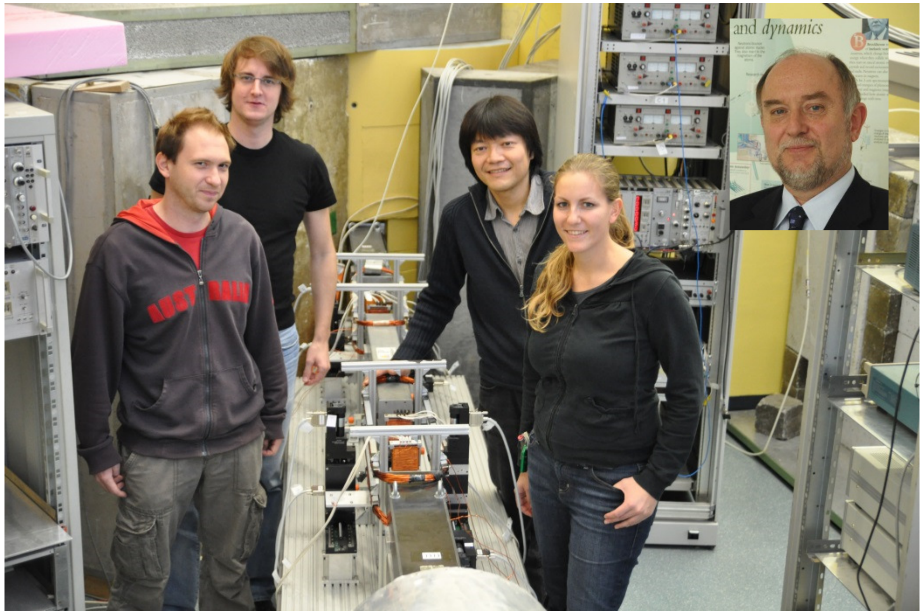
Measurement procedure

➔ operational

<quantum mechanical phenomena>

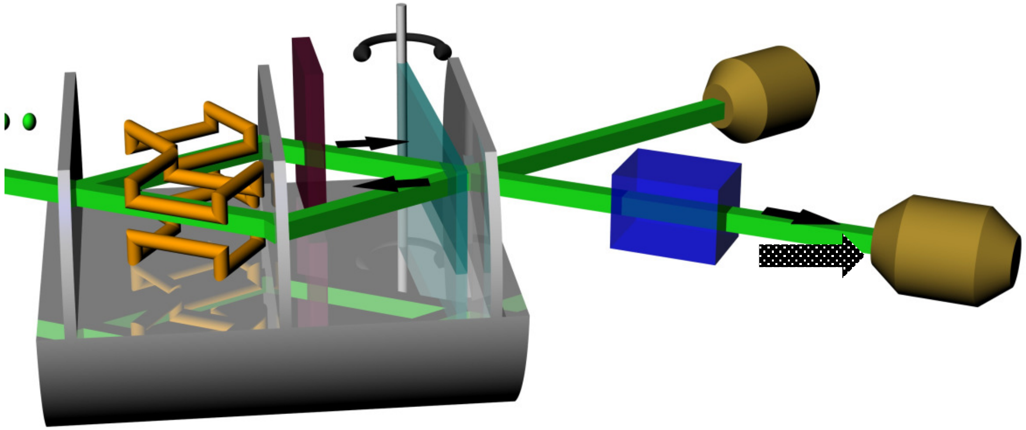
# Reactions

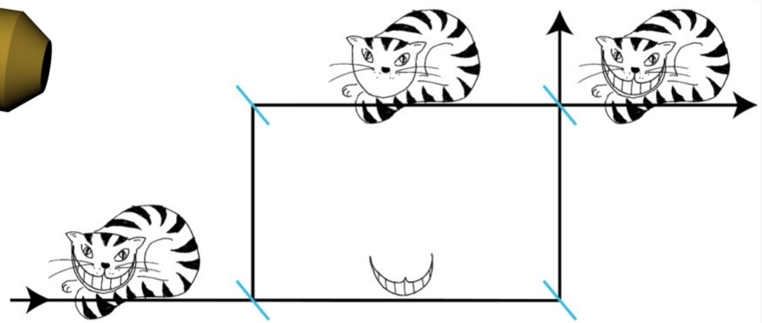

BBC
NEWS
Home | UK | Africa
29 July 2014 Last upd
'Quantun
By James Morgai
Science reporter, BBC
The Cheshire Cat myste
DER STANDARD
FORSCHUNG SPEZIAL
MITTWOCH, 30. JULI 2014
Die Katze verschwindet, ihr Grinsen bleibt
Wiener Experiment
Die Presse SAMSTAG, 16. AUGUST 2014
WISSEN
Wie Krankheiten den Lauf der Geschichte ändern Seite 54
Versteckte Emissionen eines lange verbotenen Treibhausgases Seite 55
Eine übereifrige Qualitätskontrolle, die zu Muskelschwund führt Seite 55
Ein alter Stern war Zeuge einer heftigen Explosion Seite 55
Von grinsenden Katzen und nackten Neutronen
Mithilfe schwacher Messungen schaffen Forscher im Labor das Analogon einer Cheshire Cat
Die Cheshire-Katze und ihre quantenmechanische Entsprechung
Christian Speicher
Getrennte Wege
Lust auf Salz
Schadet ein geringer Salzkonsum?
Nicola von Lutterotti
Unerwartetes Ergebnis
-Wunderland
von ihren Eigenschaften zu trennen.
im Wunderland".

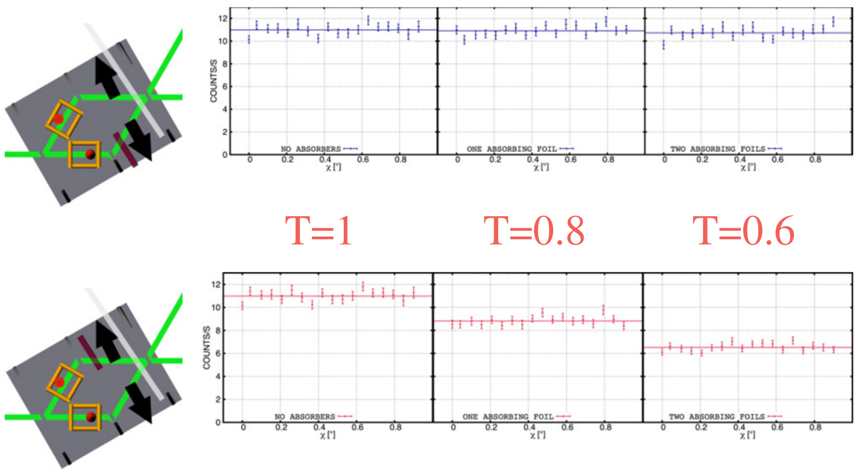


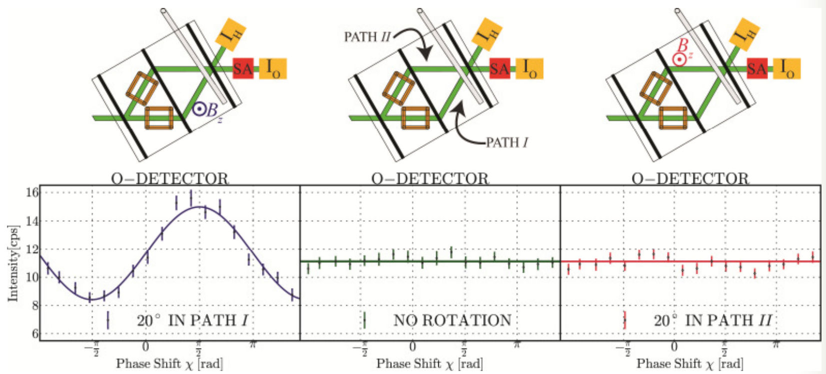

Thank you very much for your attention

# Quantum Cheshire-cat: final results







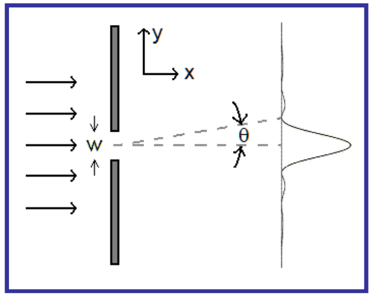



# Uncertainty relation: historical 2

- Kennard considered the spread of a wave function ψ

$$\sigma(Q)\sigma(P) \geq \frac{\hbar}{2}$$

σ: standard deviations




- Robertson generalized the relation to arbitrary pairs of observables in any states ψ

$$\sigma(A)\ \sigma(B) \geq \frac{1}{2}\left|\langle\psi|[A,B]|\psi\rangle\right|$$

→ dependent on the state but independent of the appartus

*Is* $\varepsilon(A)\eta(B) \geq \frac{1}{2}\left|\langle\psi|[A,B]|\psi\rangle\right|$ *generally valid?*

# Definition of error & disturbance

- **Error** is defined as the root-mean-square (rms):

$$\epsilon(A) = \langle\psi| \otimes \langle\xi|\big(U^\dagger(I \otimes M)U - A \otimes I\big)^2|\xi\rangle \otimes |\psi\rangle^{1/2}$$

describes how accurate the value of the observable $A$ before the measurement is transferred to the apparatus's meter observable $M$

meter observable $M$ *has* orthonormal basis $|\lambda\rangle$ : $M = \sum_\lambda \lambda|\lambda\rangle\langle\lambda|$

family of measurement ***operators***: $O_\lambda = \langle\lambda|U|\xi\rangle$ acting on object-system $\mathcal{H}^{\mathrm{obj}}$ ($U$ on $\mathcal{H}^{\mathrm{obj}} \otimes \mathcal{H}^{\mathrm{app}}$)

- $\epsilon(A)^2 = \sum_\lambda ||O_\lambda(\lambda - A)|\psi\rangle||^2$

$||...|| = ||X|\psi\rangle|| = \langle\psi|X^\dagger X|\psi\rangle^{\frac{1}{2}}$

- **Disturbance** is defined in the same manner:

$$\eta(B) = \langle\psi| \otimes \langle\xi|\big(U^\dagger(B \otimes I)U - B \otimes I\big)^2|\xi\rangle \otimes |\psi\rangle^{1/2}$$

defined by the rms difference between the observable $B$ at time $t = 0$ and at time $t = \Delta t$

- $\eta(B)^2 = \sum ||[O_\lambda, B]|\psi\rangle||^2$

# Universally valid uncertainty relation by Ozawa 1

Joint measurement of A and B in state ψ:

- $M^{out} = A^{in} + N(A)$
- $B^{out} = B^{in} + D(A)$

We obtain the following commutation relation for noise and disturbance operator with $[M^{out}, B^{out}]=0$

$$[N(A), D(B)] + [N(A), B^{in}] + [A^{in}, D(B)] \geq -[A^{in}, B^{in}]$$

applying the triangular inequality

$$|\langle[N(A), D(B)]\rangle| + |\langle[N(A), B^{in}]\rangle + \langle[A^{in}, D(B)]\rangle| \geq |Tr([A, B]\rho)|$$

M. Ozawa, Phys. Rev. A **67**, 042105 (2003).

# Error and disturbance for projective measurement

- **Error**:

$$\epsilon(A)^2 = ||\sum_\lambda O_\lambda(\lambda - A)|\psi||^2$$

If the $O_\lambda$ are mutually orthogonal projection operators sum and norm can be exchanged

$$\epsilon(A)^2 = ||(O_A - A)|\psi\rangle||^2$$

output operator: $O_A = \sum_\lambda \lambda O_\lambda$

different expression for measurement (5 expectation values):

$$\epsilon(A)^2 = \langle\psi|A^2|\psi\rangle + \langle\psi|O_A^2|\psi\rangle + \langle\psi|O_A|\psi\rangle + \underbrace{\langle\psi|AO_AA|\psi\rangle}_{\langle\psi'|O_A|\psi'\rangle} - \underbrace{\langle\psi|(A+I)O_A(A+I)|\psi\rangle}_{\langle\psi''|O_A|\psi''\rangle}$$

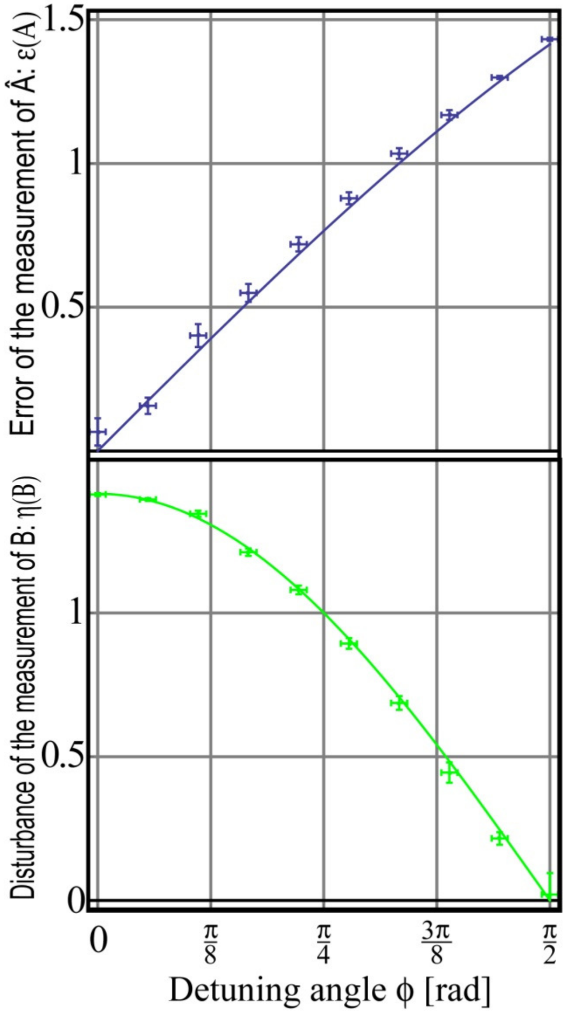
with $O_A^2 = \sum_\lambda \lambda^2 O_\lambda^\dagger O_\lambda$

- **Disturbance**: $\eta(B)^2 = \sum_\lambda ||[O_\lambda, B]|\psi\rangle||^2$

$$\eta(B)^2 = \langle\psi|B^2|\psi\rangle + \langle\psi|X_B^2|\psi\rangle + \langle\psi|X_B|\psi\rangle + \overbrace{\langle\psi|BX_BB|\psi\rangle}^{\langle\psi'''|X_B|\psi'''\rangle} - \overbrace{\langle\psi|(B+I)X_B(B+I)|\psi\rangle}^{\langle\psi''''|X_B|\psi''''\rangle}$$

with $X_B^2 = \sum_\lambda O_\lambda^\dagger B^2 O_\lambda$ , and modified output operator: $X_B = \sum_\lambda O_\lambda^\dagger B O_\lambda$

# Results: error-disturbance trade-off




$$|\psi_i\rangle = |+z\rangle$$

$$\hat{A} = \hat{\sigma_x} \qquad \hat{O}_A = \hat{\sigma}_\phi = \cos(\phi)\hat{\sigma}_x + \sin(\phi)\hat{\sigma}_y$$

$$\hat{B} = \hat{\sigma_y}$$

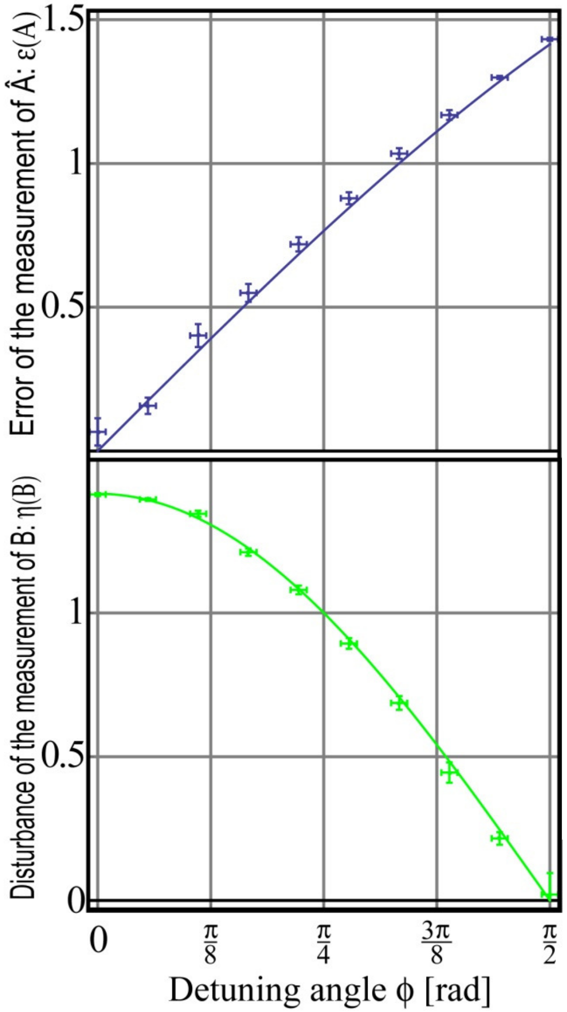
$$\varepsilon(A)^2 = \langle\psi|A^2|\psi\rangle + \langle\psi|O_A^2|\psi\rangle + \langle\psi|O_A|\psi\rangle + \langle A\psi|O_A|A\psi\rangle - \langle(A+I)\psi|O_A|(A+I)\psi\rangle$$

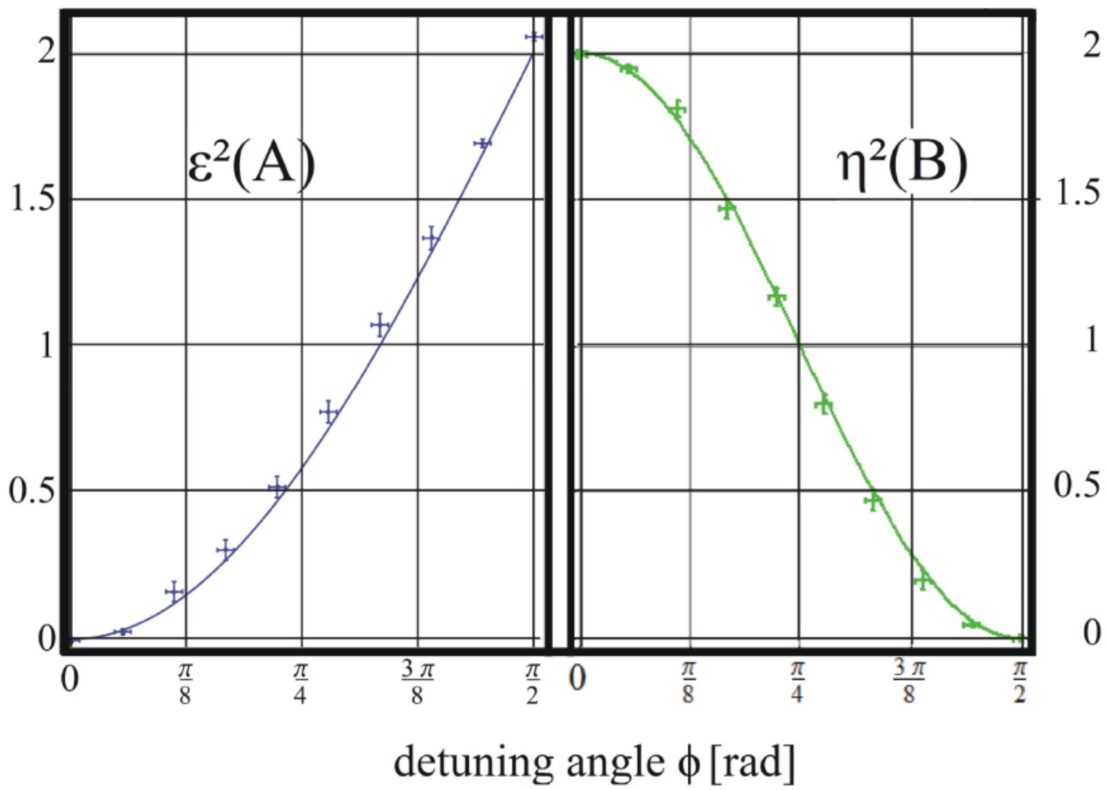
$$|\psi\rangle = |+z\rangle \qquad |\psi\rangle = |+z\rangle$$

$$|A\psi\rangle = |-z\rangle \qquad |B\psi\rangle = |-z\rangle$$

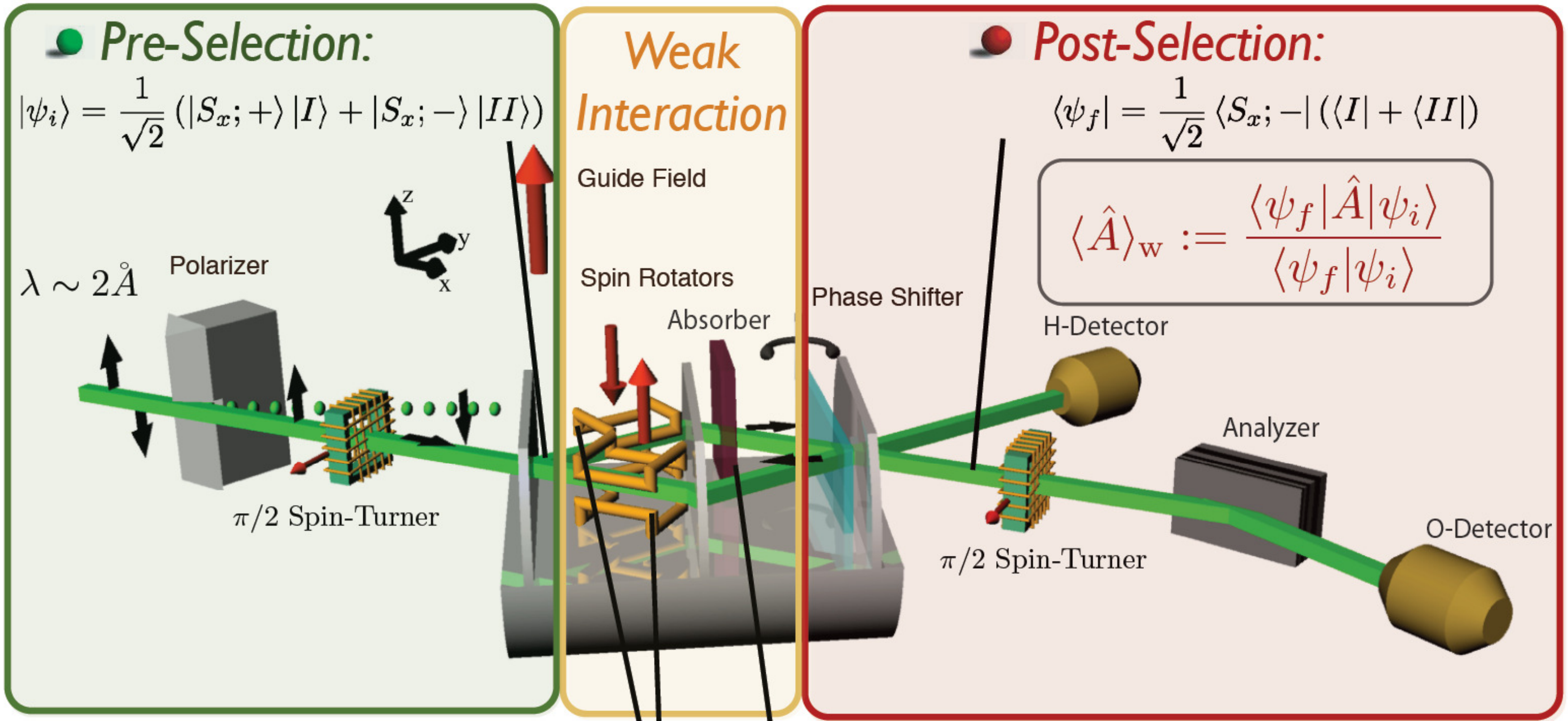
$$|(A+\mathbb{I})\psi\rangle = |+x\rangle \qquad |(B+\mathbb{I})\psi\rangle = |+y\rangle$$

# Results: error-disturbance trade-off



# Experimental setup




*imaginary optical potential* $\hat{V}_j = -i\mu_j(r)\hat{\Pi}_j$ *(path j)*

*path conditioned interaction Hamiltonian* $\hat{H}_j = -\gamma\frac{\hat{\sigma}_z}{2}B_z\hat{\Pi}_j$

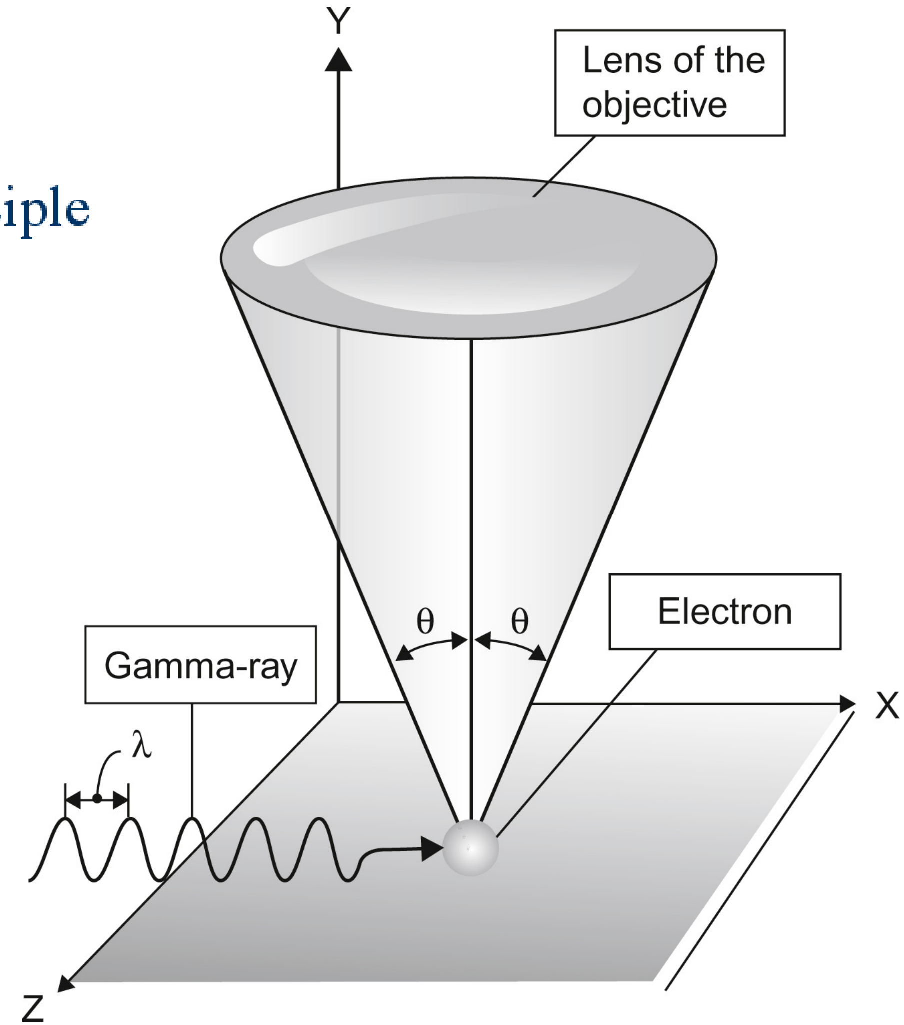
# Heisenberg's uncertainty relation

## Heisenberg's Uncertainty Principle

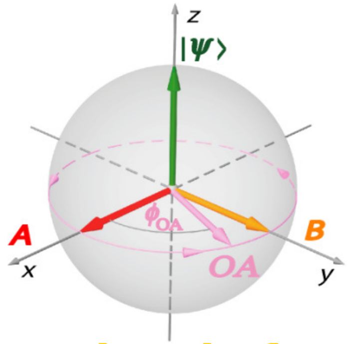
*Change in momentum*

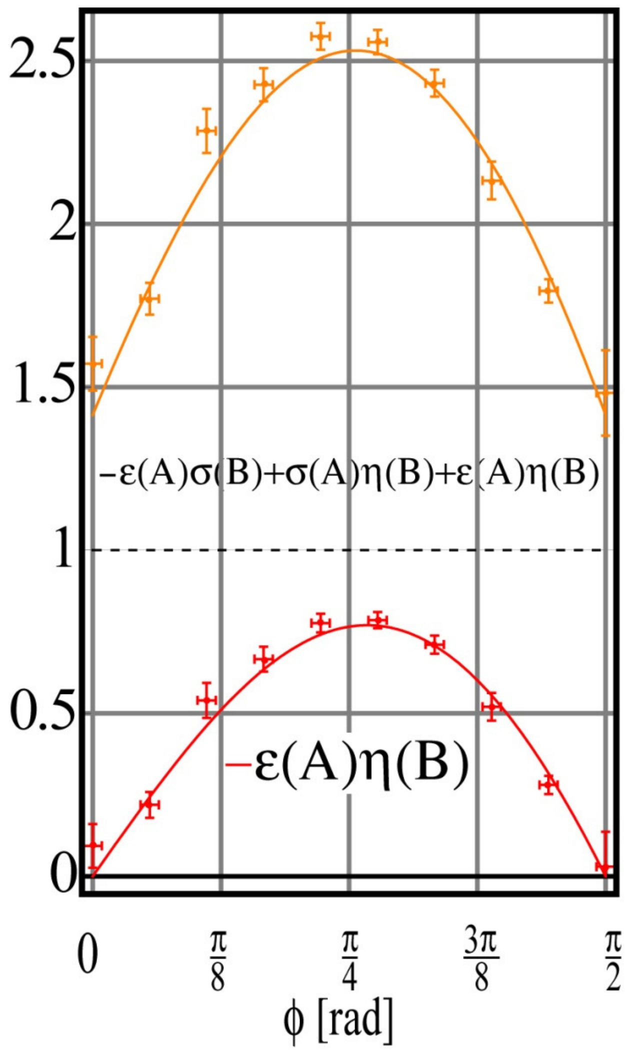
$$\Delta x \Delta p_x \geq h$$

*Change in position*



# Results: new/old uncertainty relation




New uncertainty principle

$$\begin{cases} \varepsilon : \text{error of the first measurmen} \ (A) \\ \eta : \text{disturbance on the second measurement} \ (B) \\ \sigma : \text{standard deviations} \end{cases}$$

**standard deviations:**
σ(B)= 0.9999(1)
σ(A)= 0.9994(3)

Heisenberg product

J. Erhart et al., Nature Phys. **8**, 185-189 (2012)

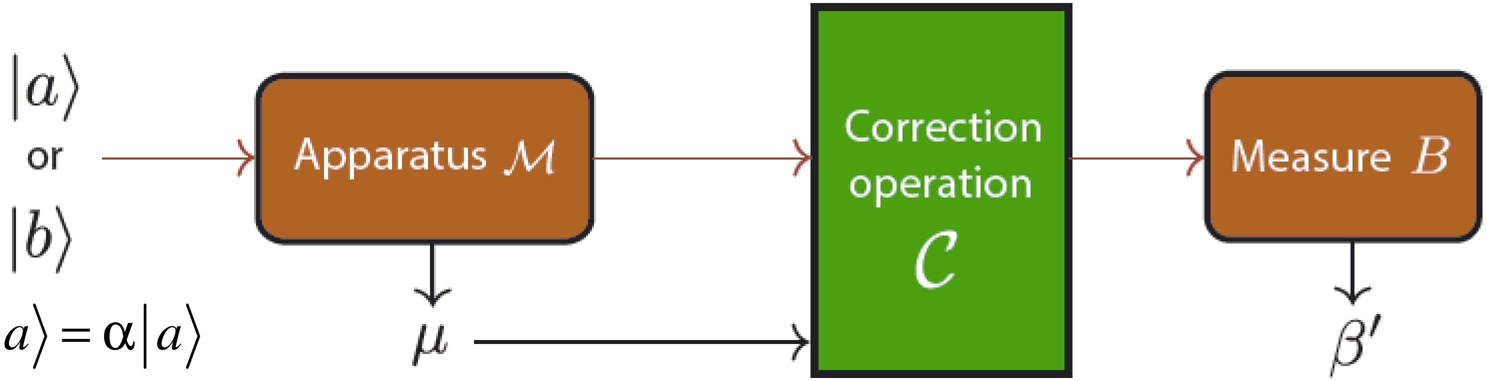
# A-error and B-disturbance: a trade-off

$$E(\mathcal{M}, A) + D(\mathcal{M}, B) \geq \log \frac{1}{c}$$

where $c = \max_{a,b} |\langle \psi_a | \varphi_b \rangle|^2$

F. Buscemi, NWW2014

# Experimental scheme 1



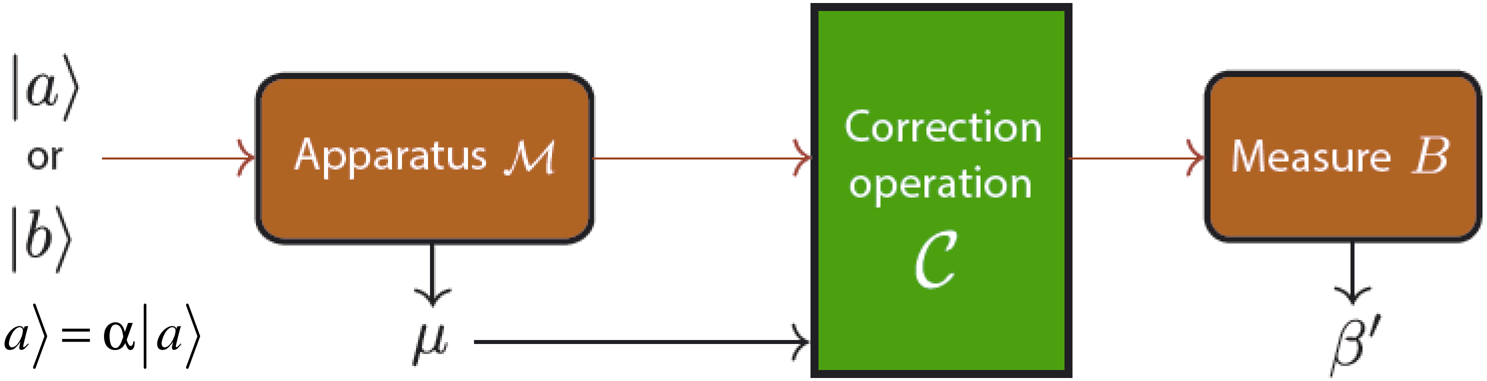

$$\begin{cases} \hat{A}|a\rangle = \alpha|a\rangle \\ \hat{B}|b\rangle = \beta|b\rangle \end{cases}$$

$$\begin{cases} N(\mathrm{M},\hat{A}) \equiv -\sum_{\alpha,\mu} p(\mu)\ \ p(\alpha \mid \mu)\ \log p(\alpha \mid \mu) = H(\mathrm{A} \mid \mathrm{M}) \\ D(\mathrm{M},\hat{B}) \equiv -\sum_{\beta,\beta'} p(\beta')\, p(\beta \mid \beta')\log p(\beta \mid \beta') = H(\mathrm{B} \mid \mathrm{B}') \end{cases}$$

*where* $p(\ldots \mid \ldots)$ : conditional probability & $p(\ldots)$ : marginal probability

$H(\ldots \mid \ldots)$ : conditional entropy

# Experimental scheme 2




$$\begin{cases} \hat{A}|a\rangle = \alpha|a\rangle \\ \hat{B}|b\rangle = \beta|b\rangle \end{cases}$$

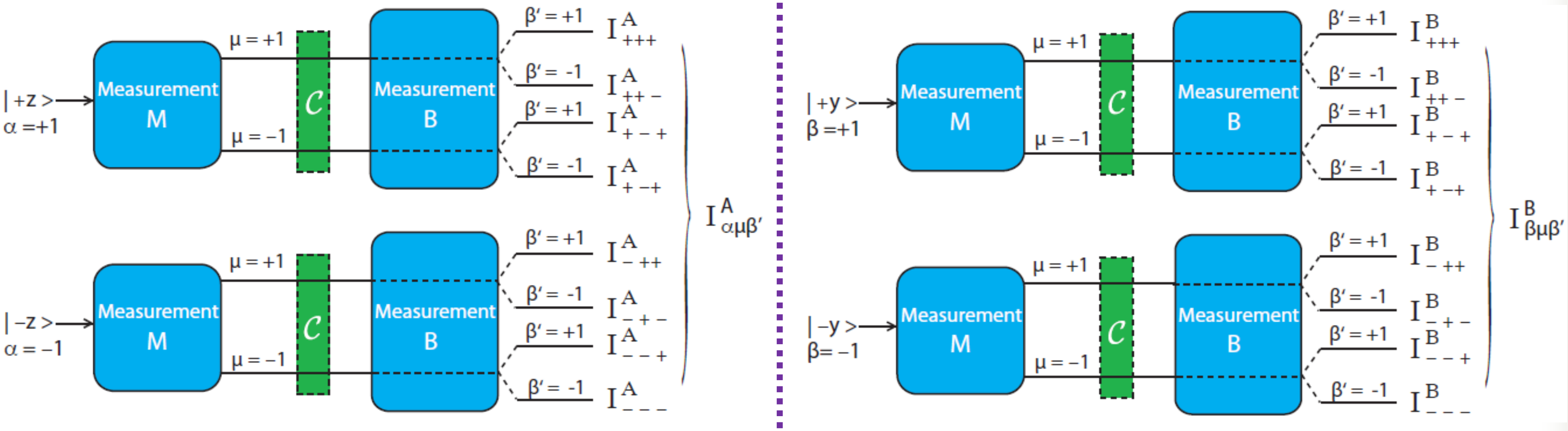
$$N(\mathbb{M},\hat{A}) + D(\mathbb{M},\hat{B}) \geq c_{AB} \equiv -\log\max|\langle a|b\rangle|^2$$

in particular, for two-level systems $\hat{A} = \hat{\sigma}_z$ & $\hat{B} = \hat{\sigma}_y$

$$g\left[N(\mathbb{M},\hat{\sigma}_z)\right]^2 + g\left[D(\mathbb{M},\hat{\sigma}_y)\right]^2 \leq 1$$

where $g[\ldots]$ : inverse function of $h(x) \equiv -\frac{1+x}{2}\log\frac{1+x}{2} - \frac{1-x}{2}\log\frac{1-x}{2}$

# Determination of Noise/Disturbance



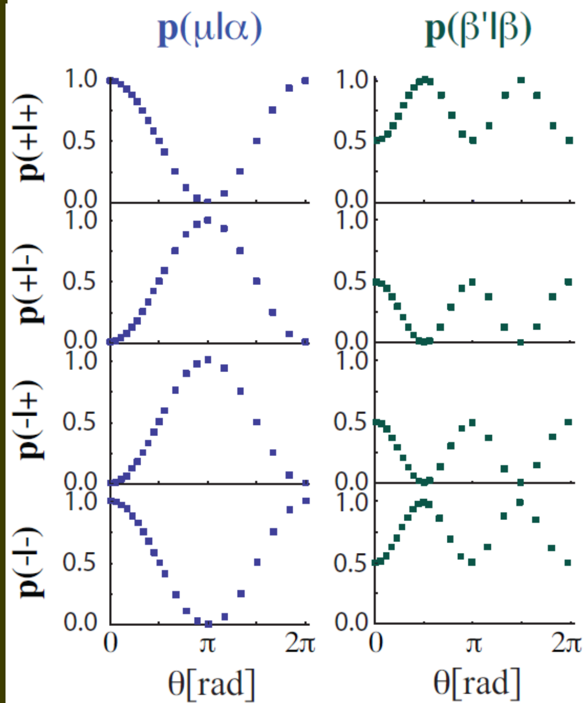

$$p(\alpha) = \frac{\sum_{\mu,\beta'} I^A_{\alpha\mu\beta'}}{\sum_{\alpha,\mu,\beta'} I^A_{\alpha\mu\beta'}} \qquad p(\mu|\alpha) = \frac{\sum_{\beta'} I^A_{\alpha\mu\beta'}}{\sum_{\mu,\beta'} I^A_{\alpha\mu\beta'}}$$

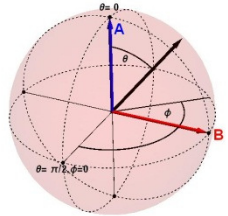
$$p(\beta) = \frac{\sum_{\mu,\beta'} I^B_{\beta\mu\beta'}}{\sum_{\beta,\mu,\beta'} I^B_{\beta\mu\beta'}} \qquad p(\beta'|\beta) = \frac{\sum_{\mu} I^B_{\beta\mu\beta'}}{\sum_{\mu,\beta'} I^B_{\beta\mu\beta'}}$$

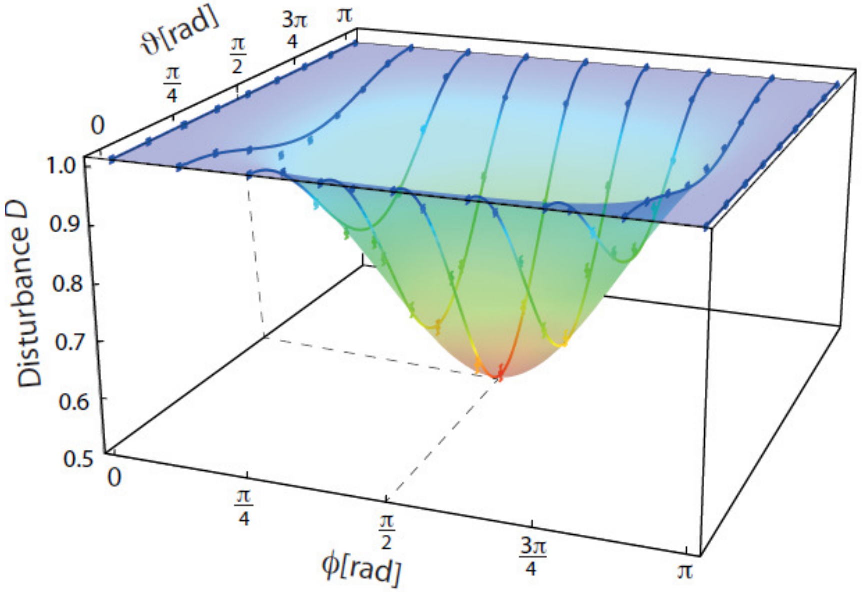
$$N(\mathrm{M},\hat{A}) \equiv -\sum_{\alpha,\mu} p(\mu)\, p(\alpha \mid \mu) \log p(\alpha \mid \mu) = H(\mathrm{A} \mid \mathrm{M})$$

$$D(\mathrm{M},\hat{B}) \equiv -\sum_{\beta,\beta'} p(\beta')\, p(\beta \mid \beta') \log p(\beta \mid \beta') = H(\mathrm{B} \mid \mathrm{B}')$$

# Experimental results:




$$p(\alpha) = p(\beta) = p_{\mathrm{opt}}(\beta) = \frac{1}{2}, \qquad p(\mu|\alpha) = \frac{1+\mu\alpha\cos\theta}{2},$$

$$p(\beta'|\beta) = p(\beta|\beta') = \frac{1+\beta'\beta\sin^2\theta}{2}, \quad p_{\mathrm{opt}}(\beta'|\beta) = p_{\mathrm{opt}}(\beta|\beta') = \frac{1+\beta'\beta|\sin\theta|}{2}$$

$$N(\mathcal{M}, A) = h(\cos\theta), \quad D_0(\mathcal{M}, B) = h\left(\sin^2\theta\right) \quad D_{\mathrm{opt}}(\mathcal{M}, B) = h(\sin\theta)$$

# Entropic Noise/Disturbance: theory

Information-theoretic formulation of Noise-Disturbance Uncertainty Relation

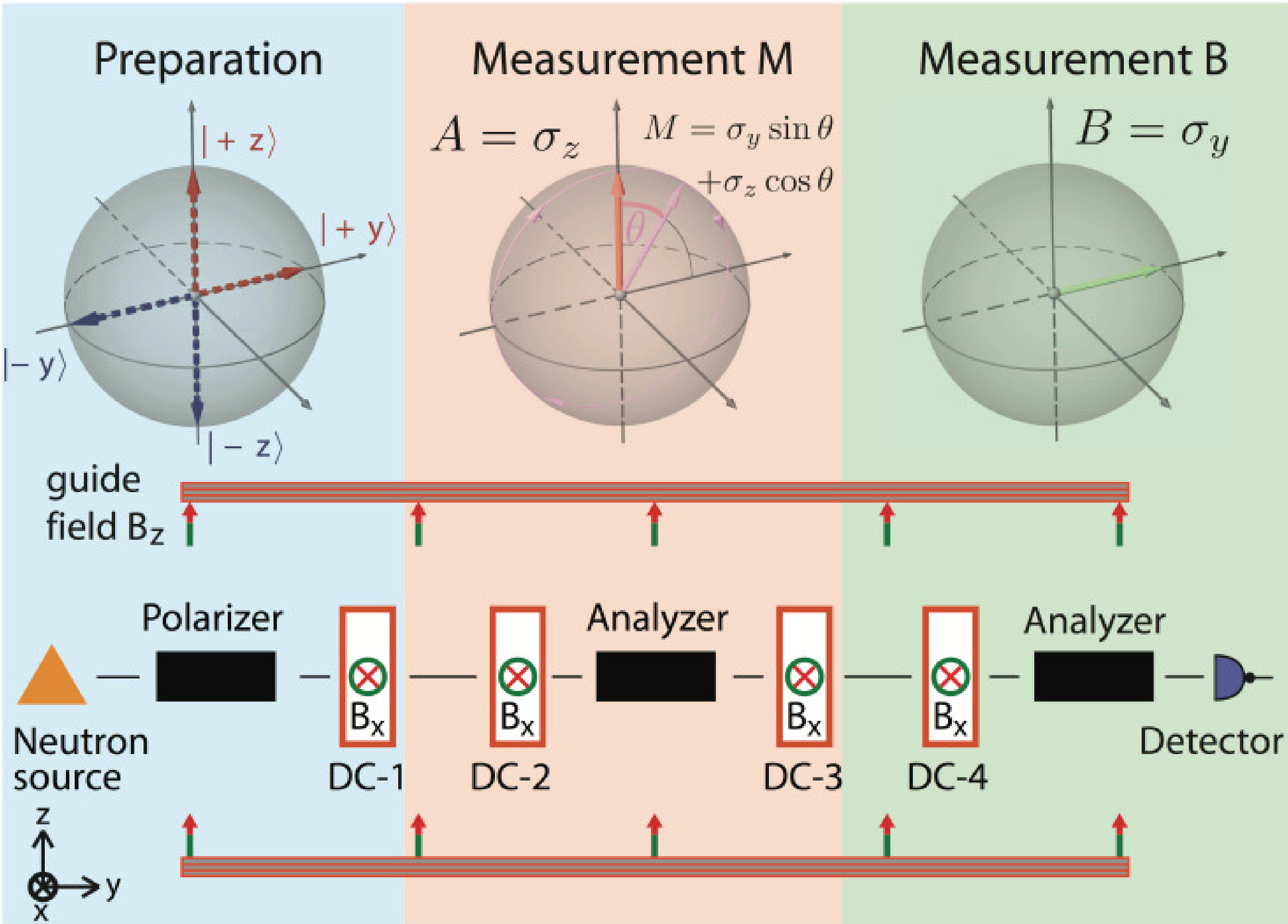
$$\boldsymbol{N}(\boldsymbol{M},\boldsymbol{A}) + \boldsymbol{D}(\boldsymbol{M},\boldsymbol{B}) \;\geq\; -\mathbf{log}(\boldsymbol{c})$$

where

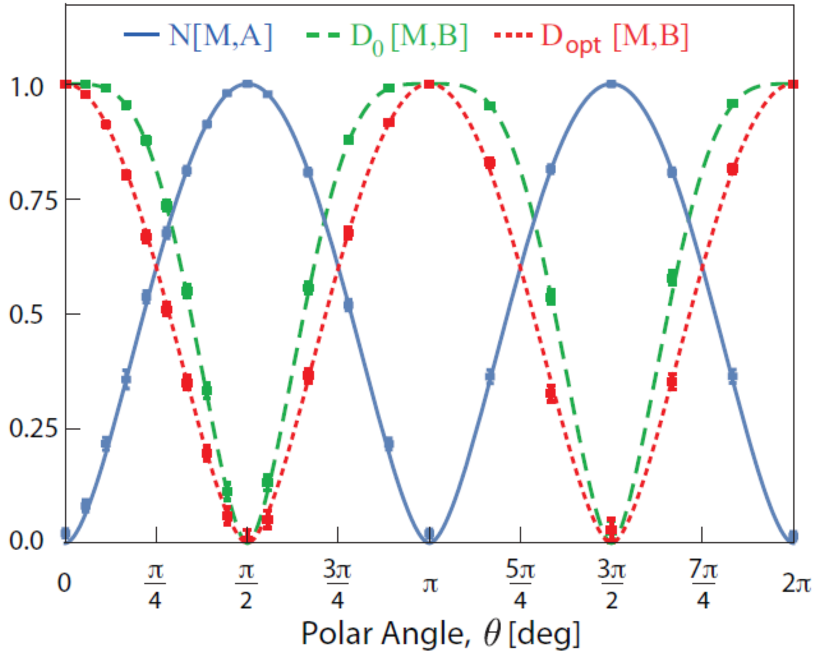
$$\boldsymbol{N}(\boldsymbol{M},\boldsymbol{A}) := \boldsymbol{H}(\boldsymbol{\mathcal{A}}|\boldsymbol{\mathcal{M}}) \;\&\; \boldsymbol{D}(\boldsymbol{M},\boldsymbol{B}) := \boldsymbol{H}(\boldsymbol{\mathfrak{B}}|\boldsymbol{\mathcal{M}})$$

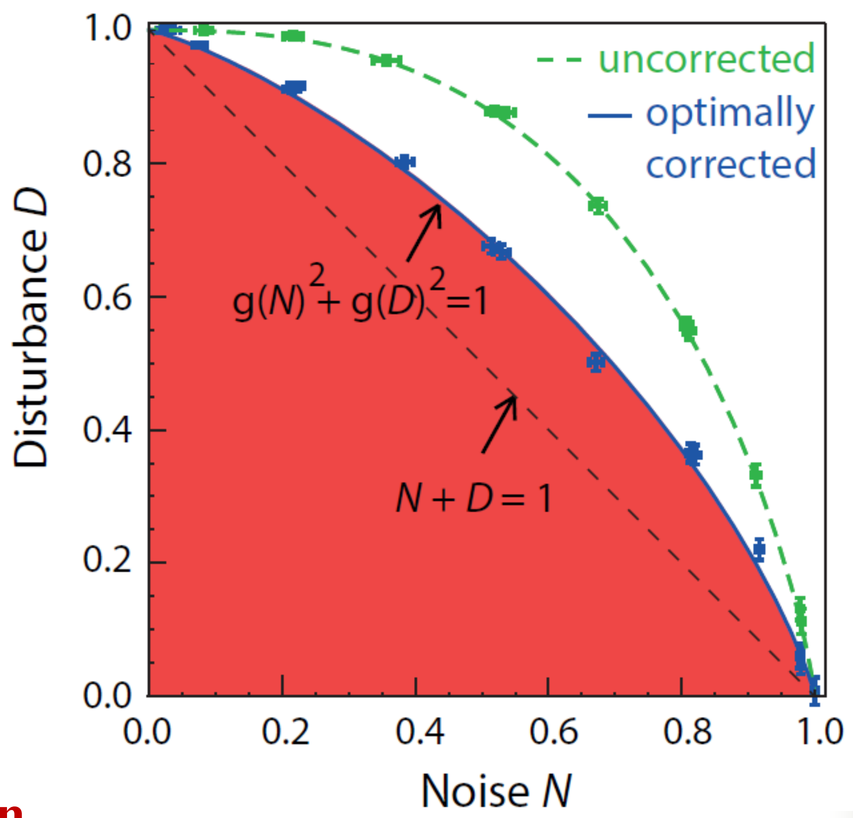
$$\boldsymbol{H}(\boldsymbol{Y}|\boldsymbol{X}) := -\sum_{x,y} \mathbf{p}(\mathbf{y},\mathbf{x})\mathbf{log}\left(\mathbf{p}(\boldsymbol{y}|\mathbf{x})\right) \text{ for joint probability}$$

# Experimental setup



# Final results for entropic noise-dist. relation




**Entropic noise-dist. uncertainty relation has π-periodicity !!!**

**Tight relation is attained.**