Quantum features of natural cellular automata

Hans-Thomas Elze

Università di Pisa

Is there a reason for the *linearity* of QM?

CellularAutomaton picture of QM $\longrightarrow$ locality implies linearity

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Linearity ...

- Def.: linearity $\Longleftrightarrow$ dynamics maps states linearly on states.
- Theorem: QM is linear.

  proofs: E.P. Wigner, V. Bargmann
  – assumption: dynamics does not change $|\langle\psi'|\psi\rangle|$

  proof: T.F. Jordan
  – assumption: no influences without interactions
  " ... that the system we are considering can be described as part of a larger system without interaction with the rest of the larger system."

... we shall replace Jordan's "no influences without interactions"

in CA models of QM with three ingredients ...

- deterministic discrete mechanics – T.D. Lee *et al.*

  $\longrightarrow$ ex. minimal time l and discrete updating rules

- sampling theory for discrete structures – A. Kempf *et al.*

  $\longrightarrow$ ex. map: CA $\leftrightarrow$ continuum QM + corrections

- "oscillator representation" of QM – A. Heslot

  $\longrightarrow$ set $\psi \equiv x + ip$

Hamiltonian Cellular Automata (CA) – *"bit machines"*

- classical CA with denumerable degrees of freedom

- state described by integer valued coordinates x_n^α, τ_n and momenta p_n^α, π_n

  $\alpha \in \mathbf{N}_0$: different degrees of freedom
  $n \in \mathbf{Z}$: successive states

- finite differences, $\Delta f_n := f_n - f_{n-1}$

  $\longrightarrow$ no infinitesimals!

The CA Action Principle <•=•= Phys. Rev. A 89, 012111 (2014) .

- $\mathcal{A}_n := \Delta\tau_n(H_n + H_{n-1}) + c_n\pi_n$,

  $H_n := \frac{1}{2}S_{\alpha\beta}(p_n^\alpha p_n^\beta + x_n^\alpha x_n^\beta) + A_{\alpha\beta}p_n^\alpha x_n^\beta + R_n(x,p)$,

  const. c_n, sym. $\hat{S} \equiv \{S_{\alpha\beta}\}$, antisym. $\hat{A} \equiv \{A_{\alpha\beta}\}$, *remainder* R_n .

- integer valued action:

  $\mathcal{S} := \sum_n[(p_n^\alpha + p_{n-1}^\alpha)\Delta x_n^\alpha + (\pi_n + \pi_{n-1})\Delta\tau_n - \mathcal{A}_n]$.

- Action Principle: $\delta\mathcal{S} \stackrel{!}{=} 0 \;\Rightarrow\;$ CA updating rules,

  for $\delta g(f_n) := [g(f_n + \delta f_n) - g(f_n - \delta f_n)]/2$, $\delta f_n \in \mathbf{Z}$, arbitrary

  Remarks ... $\Longrightarrow$ $R_n \equiv 0$, only harmonic CA consistent. ??

CA equations of motion

- $\delta\mathcal{S} \stackrel{!}{=} 0 \;\Rightarrow$ finite differences e.o.m.:

$$\dot{x}^\alpha_n = \dot{\tau}_n(S_{\alpha\beta}p^\beta_n + A_{\alpha\beta}x^\beta_n)\;,$$

$$\dot{p}^\alpha_n = -\dot{\tau}_n(S_{\alpha\beta}x^\beta_n - A_{\alpha\beta}p^\beta_n)\;,$$

$$\dot{\tau}_n = c_n\;, \quad \dot{\pi}_n = \dot{H}_n\;, \quad \text{with } \dot{O}_n := O_{n+1} - O_{n-1}\;.$$

- e.o.m. time reversal invariant, $(n \mp 1, n) \to (n \pm 1)$

- $\Longrightarrow \quad \dot{\psi}^\alpha_n = -i\dot{\tau}_n H_{\alpha\beta}\psi^\beta_n\;,$ discrete "Schrödinger equation"

  with $\hat{H} := \hat{S} + i\hat{A}$, self-adjoint, $\psi^\alpha_n := x^\alpha_n + ip^\alpha_n$, CA "time" n

CA conservation laws

- Theorem:

  For any $\hat{G}$ with $[\hat{G}, \hat{H}] = 0$, ex. discrete conservation law:

  $$\psi_n^{*\alpha} G_{\alpha\beta} \dot{\psi}_n^\beta + \dot{\psi}_n^{*\alpha} G_{\alpha\beta} \psi_n^\beta = 0 \quad .$$

  For $\hat{G} = \hat{G}^\dagger$ (complex integer) $\rightarrow$ n-indep. two-point fct.:

  $$\psi_n^{*\alpha} G_{\alpha\beta} \psi_{n+1}^\beta + c.c. = \text{const} \in \mathbf{Z} \ .$$

  For $\hat{G} = \hat{1}$: $\psi_n^{*\alpha} \psi_{n+1}^\alpha + c.c. = \text{const}$. [cf. $\psi^{*\alpha}\psi^\alpha = 1$]

- Ex. 1-to-1 correspondence CA $\leftrightarrow$ QM conservation laws.

Consistent anharmonic CA <•=•= JPCS 631, 012069 (2015) .

- action with anharmonic polynomial terms – e.g., $(x_n^\alpha x_n^\alpha)^2$

  $\Longrightarrow$ consistent CA e.o.m., provided

  $\delta g_f(f) := [g(f+\delta f) - g(f-\delta f)]/2\delta f$, $\delta f \in \mathbf{Z}$, arbitrary

  generalized by

  $\delta_f g^{(N)}(f) := \sum_k \gamma_k [g^{(N)}(f+m_k\delta f) - g^{(N)}(f-m_k\delta f)]/2\delta f$,

- such that $\delta_f \;\widehat{=}\; \mathrm{d}/\mathrm{d}f$:

  $\delta_f g^{(N)}(f) = \bar{g}^{(N-1)}(f)$, terms $\propto (\delta f)^j$, $j > 0$ cancel .

  $\Longrightarrow$ discrete nonlinear "Schrödinger equation" . ??

Towards continuum QM ...

- recall $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$, CA "time" n

  introduce minimal time $l \longrightarrow n \cdot l$, physical time?

  $\Longrightarrow$ continuum limit, $l \to 0$, does not work
  – integer valuedness $\Rightarrow$ time derivatives diverge!

- construct invertible map:

  discrete (integer valued) $\longleftrightarrow$ continuous (differentiable)
  – G. 't Hooft

- simultaneously continuous & discrete information
  – C.E. Shannon

The Sampling Theorem

- Consider square integrable bandlimited functions f :

  $$f(t) = (2\pi)^{-1} \int_{-\omega_{max}}^{\omega_{max}} \mathrm{d}\omega \; \mathrm{e}^{-i\omega t} \tilde{f}(\omega) \; , \; \text{bandwidth } \omega_{max} \; .$$

- Shannon's Theorem:

  Given $\{f(t_n)\}$ for set $\{t_n\}$ of equidistantly spaced times (spacing π/ω_{max}), function f is obtained for all t by:

  $$f(t) = \sum_n f(t_n) \cdot \frac{\sin[\omega_{max}(t-t_n)]}{\omega_{max}(t-t_n)} \quad \text{(reconstruction formula)} \; .$$

- CA "time" n $\sim$ discrete time $t_n := nl$ $\rightarrow$ continuous time t
  bandwidth $\omega_{max} := \pi/l$ (Nyquist rate)

Map: discrete CA $\leftrightarrow$ continuous QM

- by Shannon's *reconstruction formula* ...
  discrete e.o.m., $\dot{\psi}^\alpha_n \;=\; -i\hat{H}_{\alpha\beta}\psi^\beta_n$,
  $\longleftrightarrow$ continuous time "Schrödinger equation":
  $$(\hat{D}_l - \hat{D}_{-l})\psi^\alpha(t) = 2\sinh(l\partial_t)\psi^\alpha(t) = -iH_{\alpha\beta}\psi^\beta(t) \;,$$
  with $\hat{D}_T f(t) := f(t+T)$.
- $\Rightarrow$ l-dependent dispersion relation & conservation laws . *
- $\Rightarrow$ $l \to 0$ reproduces corresponding QM results .
- $\Rightarrow$ different linear reconstructions $\longrightarrow$ same e.o.m. & conservation laws – wave fct. "cut-off" changes .

Note: l-dependence (continuous description)

- l-dependent constants of motion:

  from Theorem on discrete conservation laws,

  for any $\hat{G}$ with $[\hat{G},\hat{H}]=0$ and $\hat{G}=\hat{G}^\dagger$ (complex integer),

  $\Rightarrow\ \Re\big[\psi^{*\alpha}(t)G_{\alpha\beta}\psi^{\beta}(t+l)\big]=\text{const}\in\mathbf{Z}$.

- l-dependent dispersion relation:

  $\Rightarrow\ lE_\alpha=\arcsin(\frac{l\epsilon_\alpha}{2})=\frac{l\epsilon_\alpha}{2}[1+(\frac{l\epsilon_\alpha}{2})^2/6+O\big((l\epsilon_\alpha)^4\big)]$,

  where $\hat{H}\to\{l\epsilon_\alpha\}$ and, thus, $|E_\alpha|\leq\pi/2l=\omega_{max}/2$.

What goes wrong with anharmonic CA ...

- by Shannon's *reconstruction formula* ...

  discrete $\psi_n \longleftrightarrow \psi(t)$, bandlimited $\sim \pi/l$

  discrete anharmonic, e.g., $(\psi_n)^2 \longleftrightarrow \psi_{(2)}(t) =$??

- $\psi_n = l^{-1} \int \mathrm{d}t\, s_n(t)\psi(t)$, $\quad s_n(t) := \mathrm{sinc}[\pi(t/l - n)]$

- $\psi_{(2)}(t) = l^{-2} \int\int \mathrm{d}t'\mathrm{d}t'' \left[\sum_n s_n(t) s_n(t') s_n(t'') \right] \psi(t')\psi(t'')$,

  $\rightarrow$ correctly bandlimited, but nonlocal in time .

- $\Rightarrow$ anharmonic CA not describable in a local continuous way
  $\Rightarrow$ no l-dep. CA based nonlinear Schrödinger equation .

13 – CONCLUSIONS

- Common l-dependent aspects of natural CA & QM:

  eqs. of motion, conservation laws; observables, admiss. $\hat{H}$

- Map: CA $\leftrightarrow$ QM based on linear Sampling Theory

  ... fails for nonlinear CA $\longrightarrow$ alternatives ??

- *Desiderata*:

  composites with $\bigotimes$-product structure (in progress)

  random or nonunitary aspects

  relativistic/QFT extension ?

Consistent CA observables?

Discrete Poisson brackets ...

- recall: only variational derivatives for discrete variables $\delta g(f) := [g(f+\delta f) - g(f-\delta f)]/2\ ,\ \ f, \delta f \in \mathbf{Z}$.
- $\rightarrow$ define: $\{A, B\} := \delta_{x^\alpha} A\, \delta_{p^\alpha} B - \delta_{x^\alpha} B\, \delta_{p^\alpha} A$.
- for *constant, linear, or quadratic polynomials* A, B, variational derivatives independent of δ_x, δ_p and bracket corresponds to ordinary Poisson bracket, in all respects.
- $\Rightarrow$ CA observables can be chosen as real quadratic forms in $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$; a closed algebra endowed with $\{\ ,\ \}$.

E.g., $\dot{\psi}^\alpha = \{\psi^\alpha, \mathcal{H}\}$, with $\mathcal{H} := \psi^{*\alpha} H_{\alpha\beta} \psi^\beta / 2$.