TEXAS TECH UNIVERSITY

October 6, 2013
Emergent Quantum Mechanics 2013
Vienna, AUSTRIA

Trajectory-based Theory of Relativistic Quantum Particles

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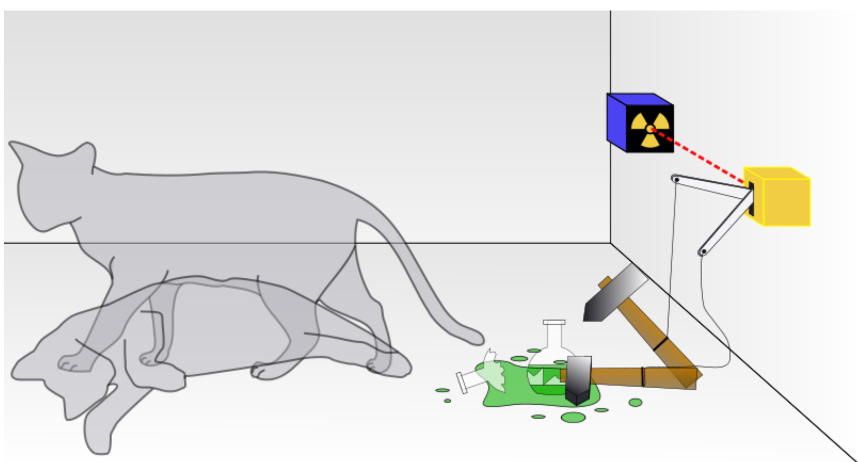
Getting Rid of Ψ Altogether:

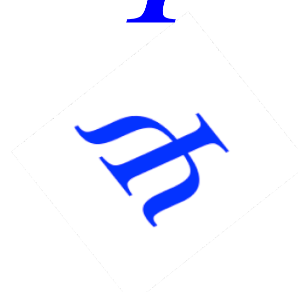
How can that even be possible?

Not

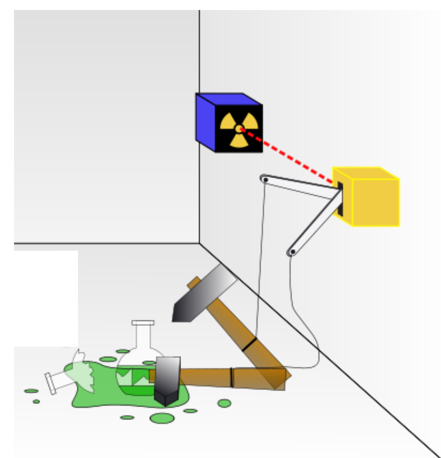


,

Ψ



But



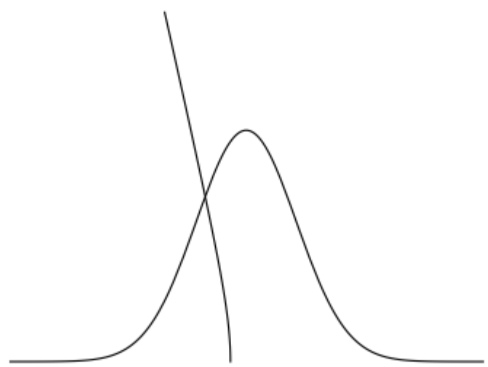
Is Ψ Alive or Dead ?

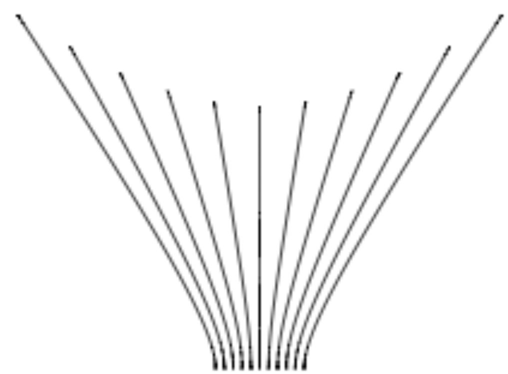
But if not Ψ, then what?

Answer: Trajectories only

- The wavefunction $\Psi(x)$ is replaced with an *ensemble* of trajectories, $x(C,t)$.
  - parameter C labels individual trajectories within the ensemble.
  - resembles classical statistical mechanics/trajectory simulations.
- The nonrelativistic individual trajectories turn out to be the quantum trajectories of David Bohm. *However…*
- *This is* **NOT** *Bohmian Mechanics!*
  - Bohm uses a *single* trajectory, $x(t)$.
  - Bohm *also* uses the wavefunction, $\Psi(x)$.

Copenhagen quantum mechanics	**Bohmian mechanics**	**Quantum trajectory-based formulation (non-relativistic)**
Ψ represents the state of the system. TDSE drives evolution of $\Psi(x,t)$.	Ψ and $x(t)$ together represent the state of the system. Ψ leads to quantum potential Q, driving trajectory dynamics via: $$m\ddot{x} + \frac{\partial V(x)}{\partial x} + \frac{\partial Q(x,t)}{\partial x} = 0$$	There is no Ψ. $x(t, C)$ (trajectory ensemble) alone represents the state of the system, and leads to Q. $x(t, C)$ satisfies its own PDE that replaces the TDSE (with $'$ denoting partial derivative w/ respect to C.) $$m\ddot{x} + \frac{\partial V(x)}{\partial x} + \frac{\hbar^2}{4m}\left(\frac{x''''}{x'^4} - 8\frac{x'''x''}{x'^5} + 10\frac{x''^3}{x'^6}\right) = 0$$





[1] A. Bouda, Int. J. Mod. Phys. A 18, 3347 (2003).
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[3] B. Poirier, Chem. Phys. 370, 4 (2010).
[4] J. Schiff and B. Poirier, J. Chem. Phys. 136, 031102 (2012).
[5] B. Poirier, arXiv:1208.6260 [quant-ph], (2012).

Broad-Ranging Ramifications of the Trajectory-Based Approach

- Theoretical/Mathematical
  - new formulation (derivation?) of quantum mechanics.
  - new conservation laws and conserved quantities.
  - new dynamical laws and kinematic forms, e.g. for Q.
- Computational/Numerical
  - direct PDE solution of $x(C,t)$ replaces Schroedinger Equation.
  - approximate ODE solution replaces classical simulations.
- Measurement/Interpretational
  - double slit exp't, EPR paradox, def'n of measurement, etc.
  - possible many-worlds-like interpretation.
  - no wavefunction means no wavefunction collapse, *per se*.

Relativistic Generalization

- Our approach is "natural," because it involves action-extremizing trajectories.
- Usual approach with Ψ-based Lagrangian leads to Klein-Gordon wave equation, which fails to give a meaningful single-particle interpretation.
  - The free-particle Klein-Gordon equation is:

$$\left(-\frac{\partial^2}{\partial (ct)^2} + \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right)\Phi(t,x) = 0 \qquad \eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$$

  - Non-physical negative-energy solutions.
  - The temporal part of the four-current density is:

$$j^0 = \frac{i\hbar}{2mc}\left(\Phi^* \frac{\partial}{\partial t}\Phi - \Phi \frac{\partial}{\partial t}\Phi^*\right)$$

  - j^0 is ***not*** positive-definite in general.
  - The four-current density j^α is ***not*** time-like in general.
- All of above issues seem to be avoided in our relativistic trajectory-based approach.

Relativistic Trajectory Theory

Motivated by Two Questions:

1. What is the Schroedinger equation the non-relativistic limit of ?

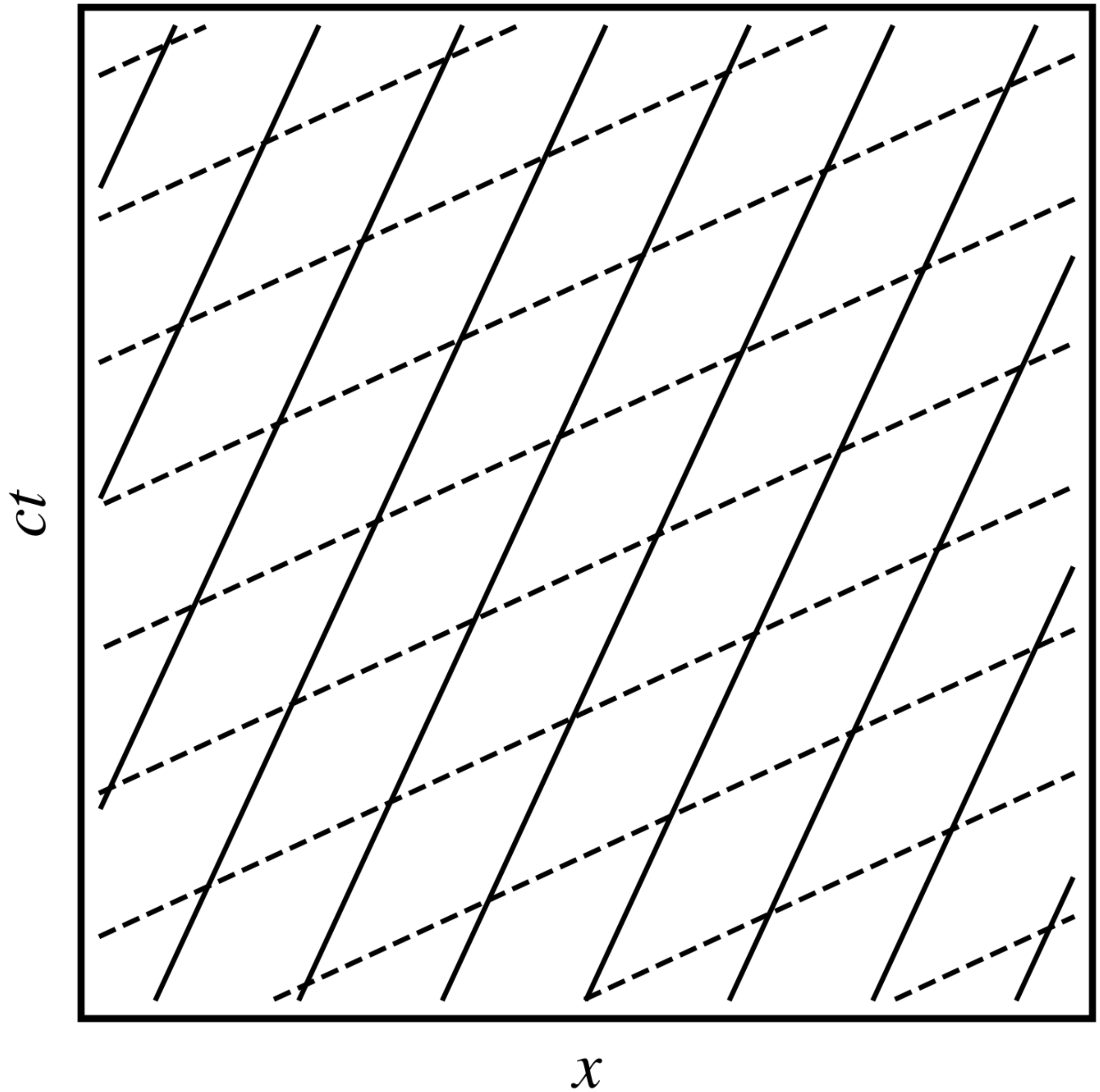
Dynamics

2. Why does relativity theory provide no notion of global *simultaneity for accelerating particles?*

Kinematics

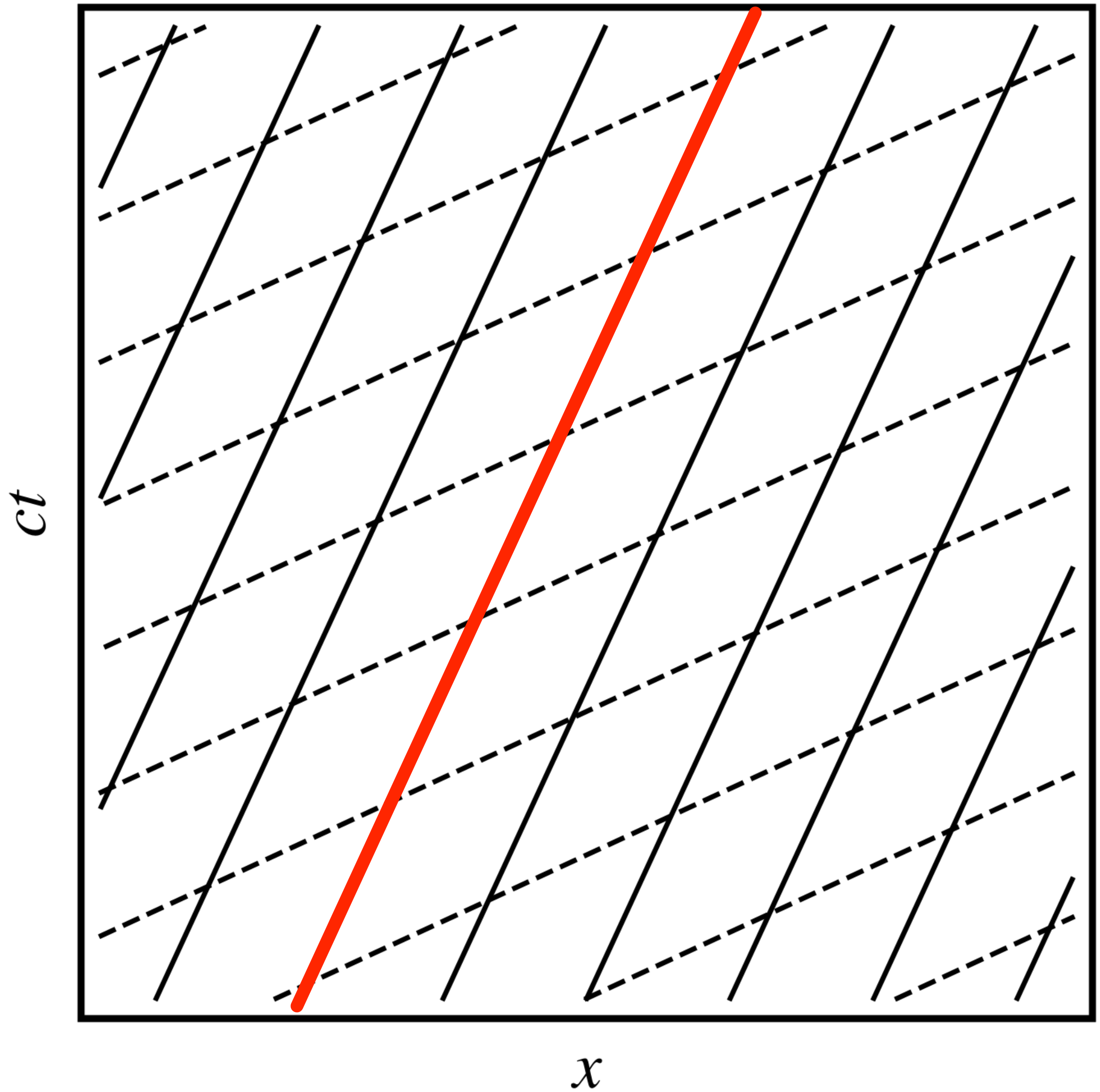
Part I:
Kinematics

Einsteinian Relativity (1+1)



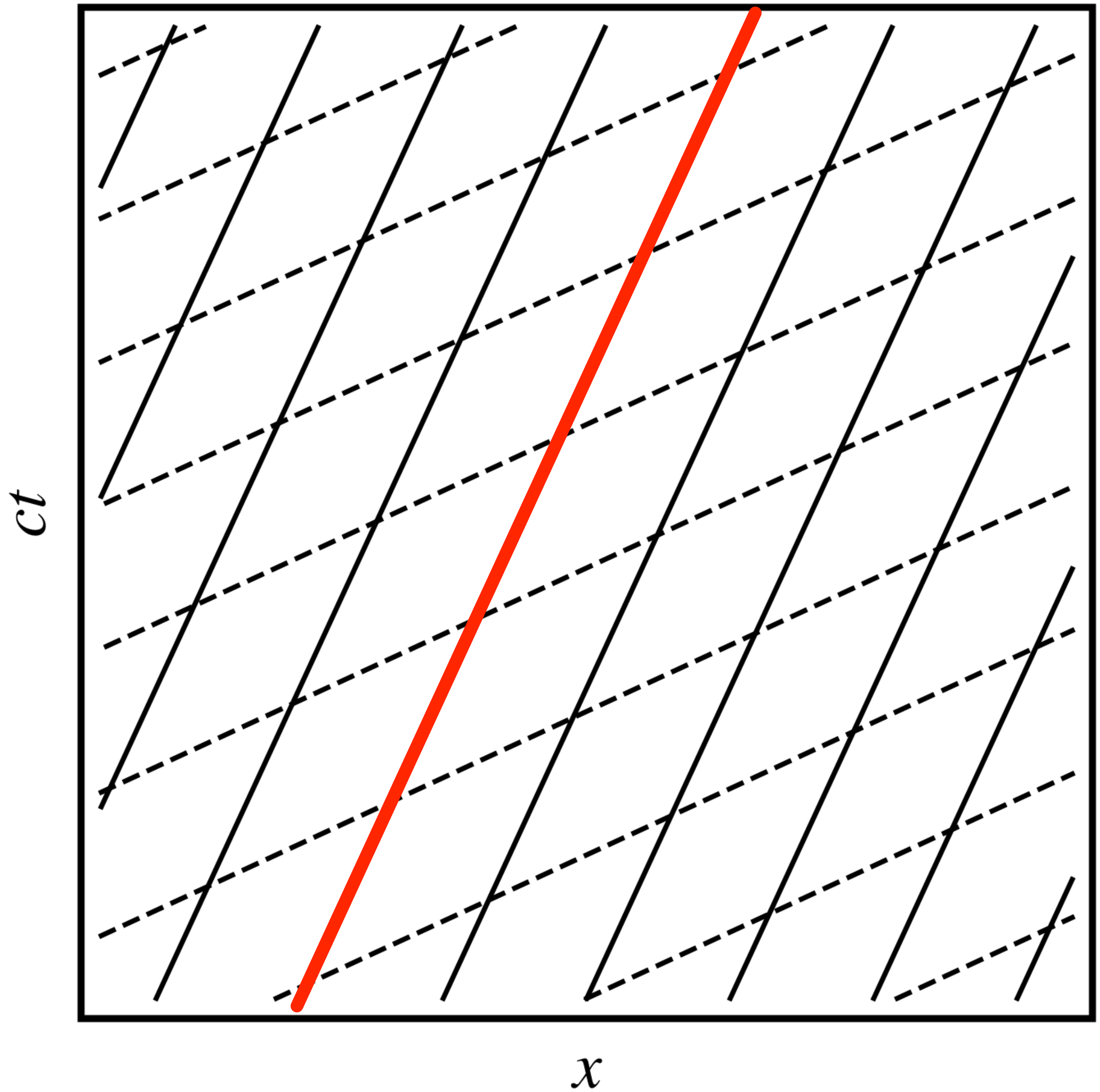
- Simultaneity well-defined for a given inertial observer, but, *depends* on observer.

Einsteinian Relativity (1+1)



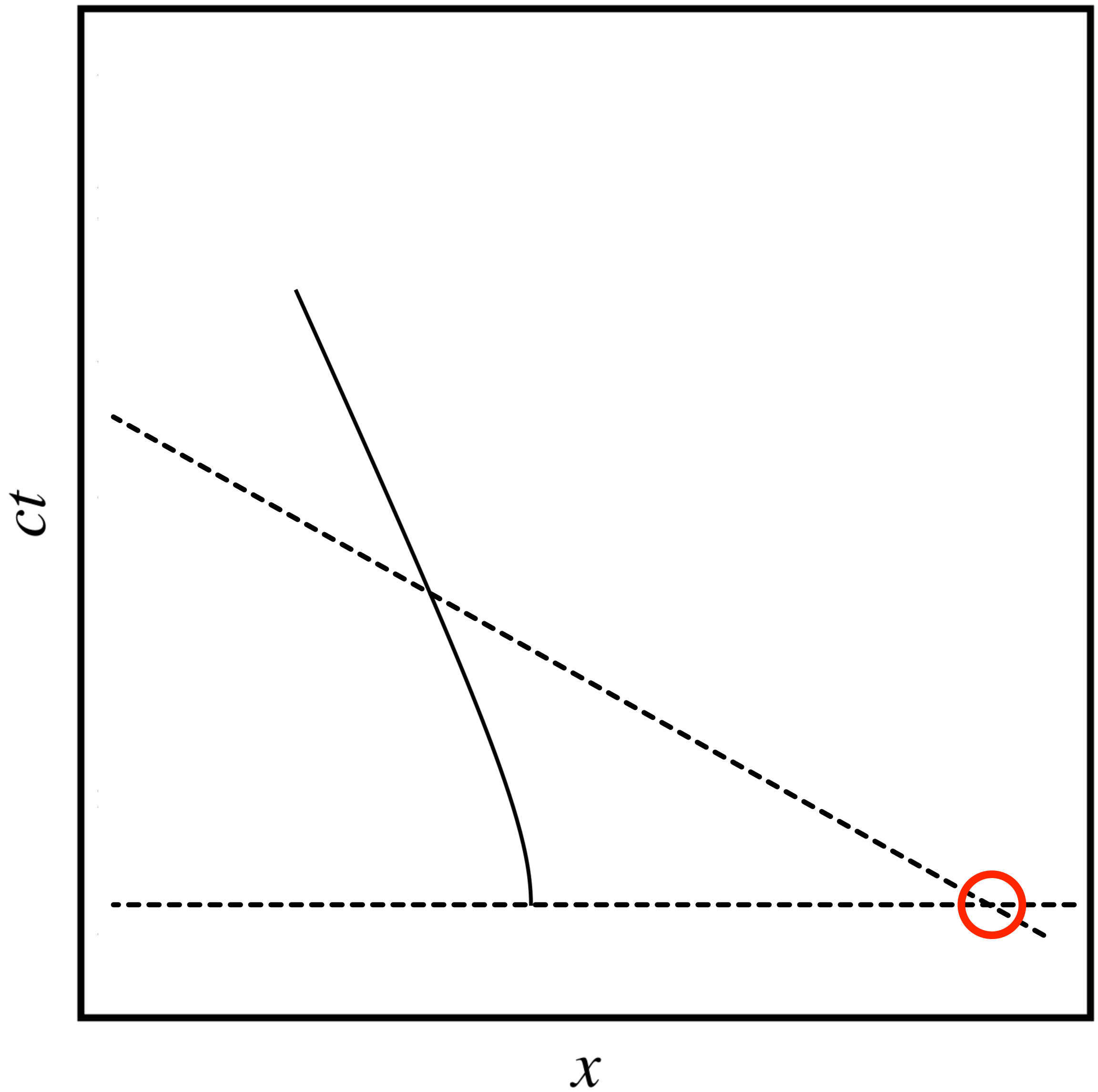
- Simultaneity well-defined for a given inertial observer, but, *depends* on observer.
- A single inertial particle (red curve) suffices to define an entire (ct', x') inertial frame (whose contours are the dashed and solid lines, respectively.)

Einsteinian Relativity (1+1)



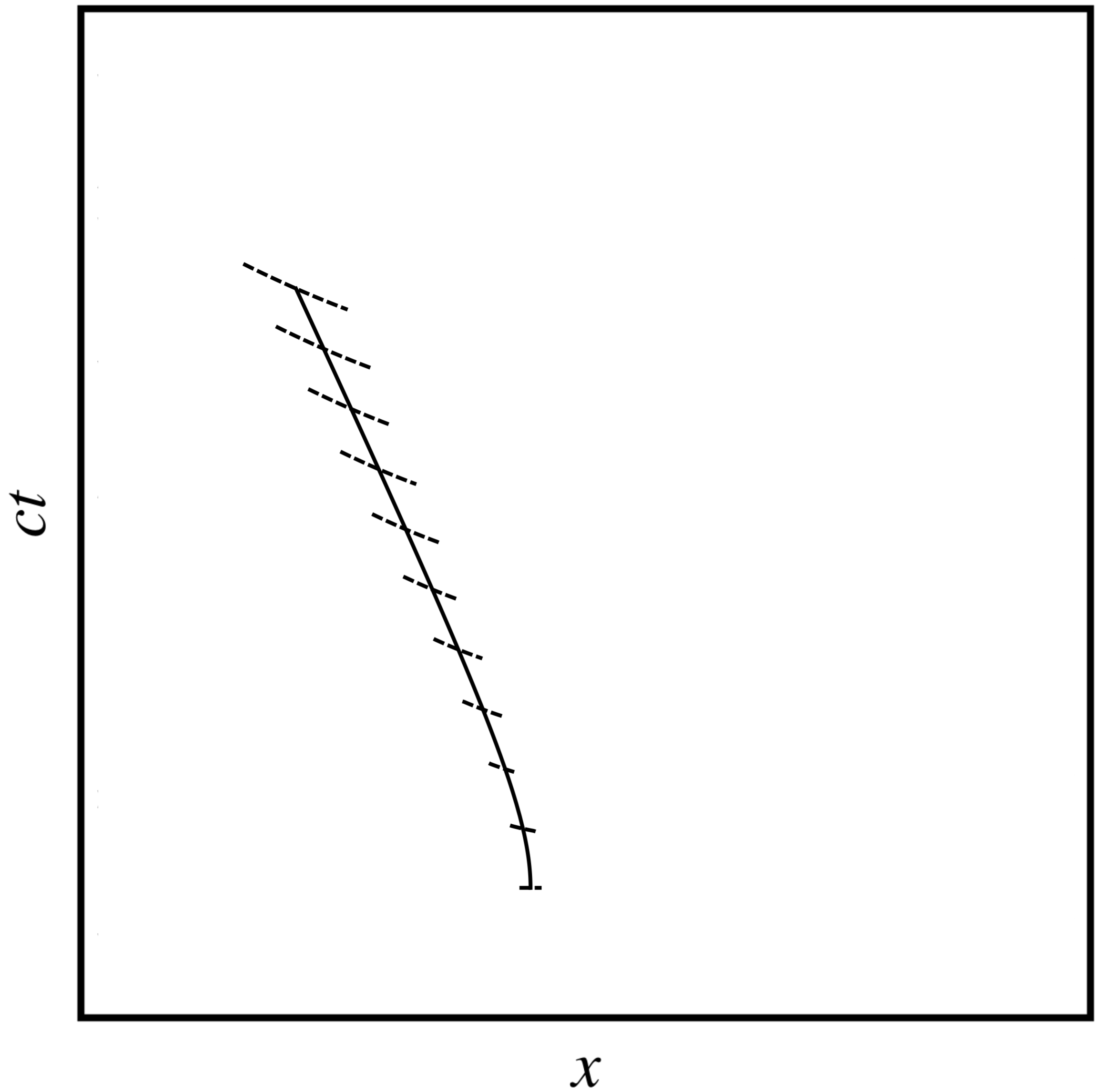
- Simultaneity well-defined for a given inertial observer, but, *depends* on observer.
- Simultaneity *not globally defined* for accelerating particles, *no matter how gentle the acceleration.*

Simultaneity for Accelerating Particles (1+1)



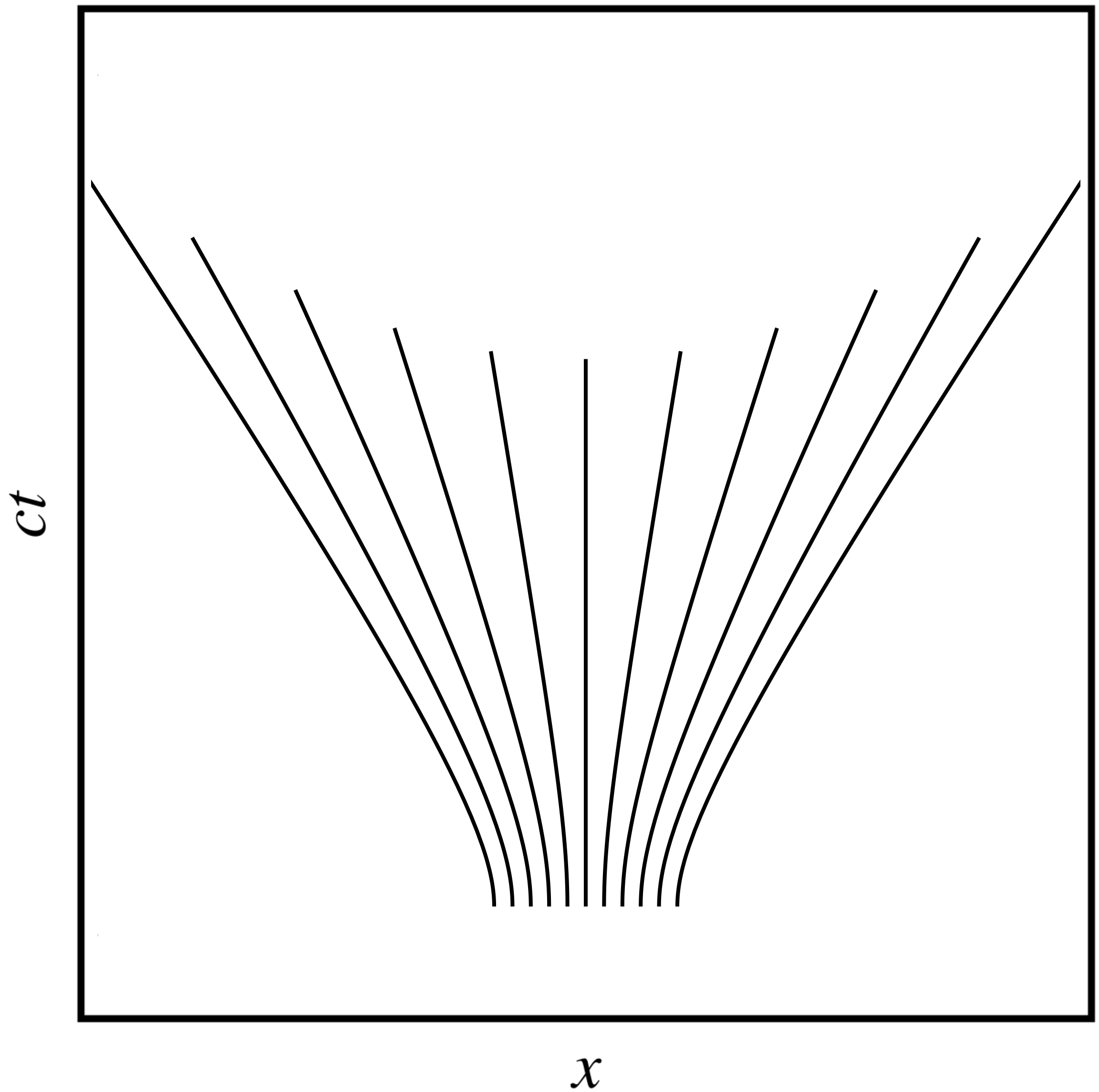
- Solid curve = worldline for an accelerating observer.
- Comoving frame approach: straight line extrapolation of local simultaneity.
- ***Problem***: simultaneity submanifolds cross! Global simultaneity is ill-defined.

Simultaneity for Accelerating Particles (1+1)



• According to relativity theory, *local* simultaneity can be defined for accelerating particles, but *global* simultaneity can not.

Simultaneity for Accelerating *Quantum* Particles (1+1)



• The system now consists of an *ensemble* of quantum trajectories.

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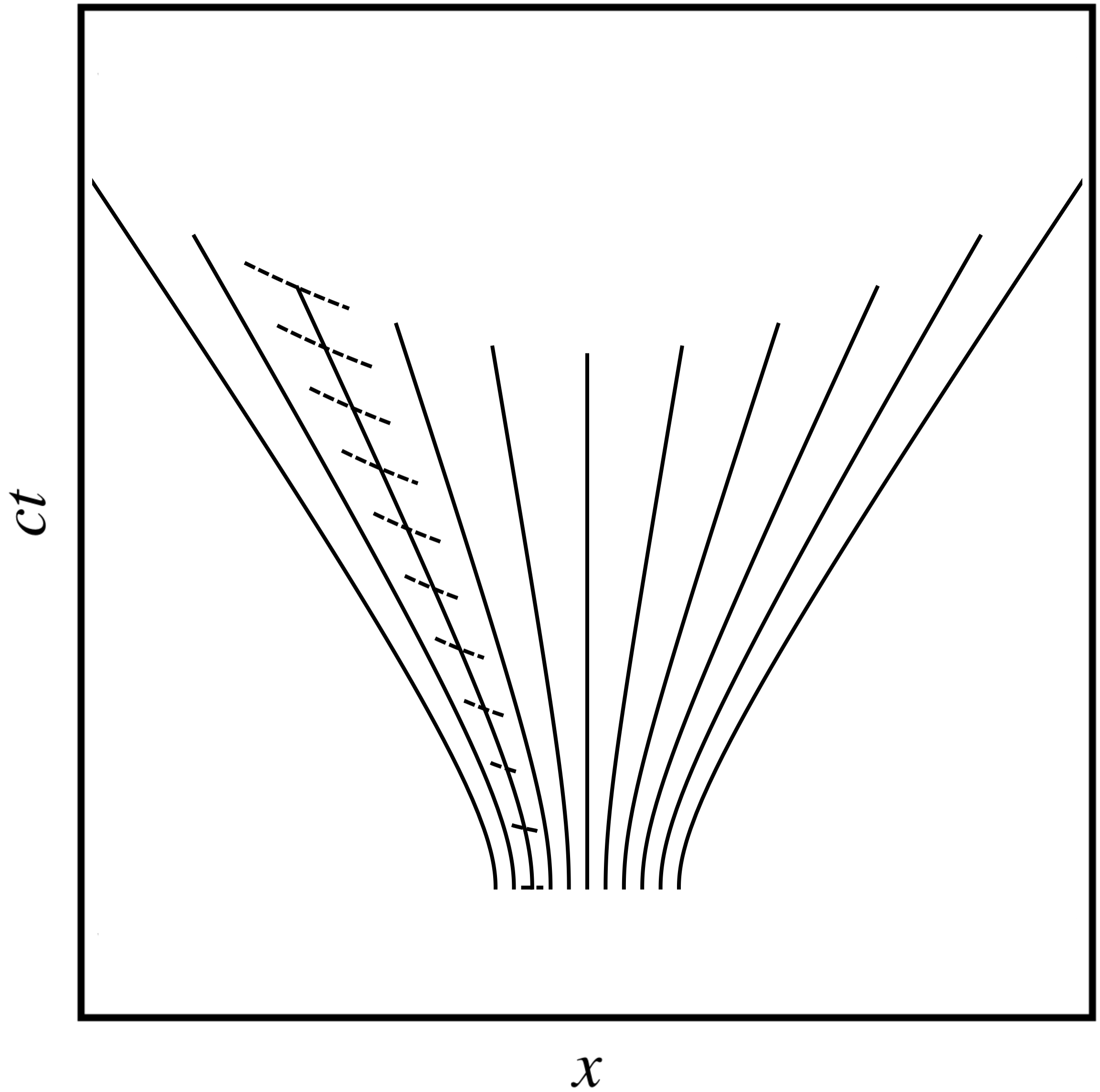

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- Each trajectory has its own local simultaneity segments.

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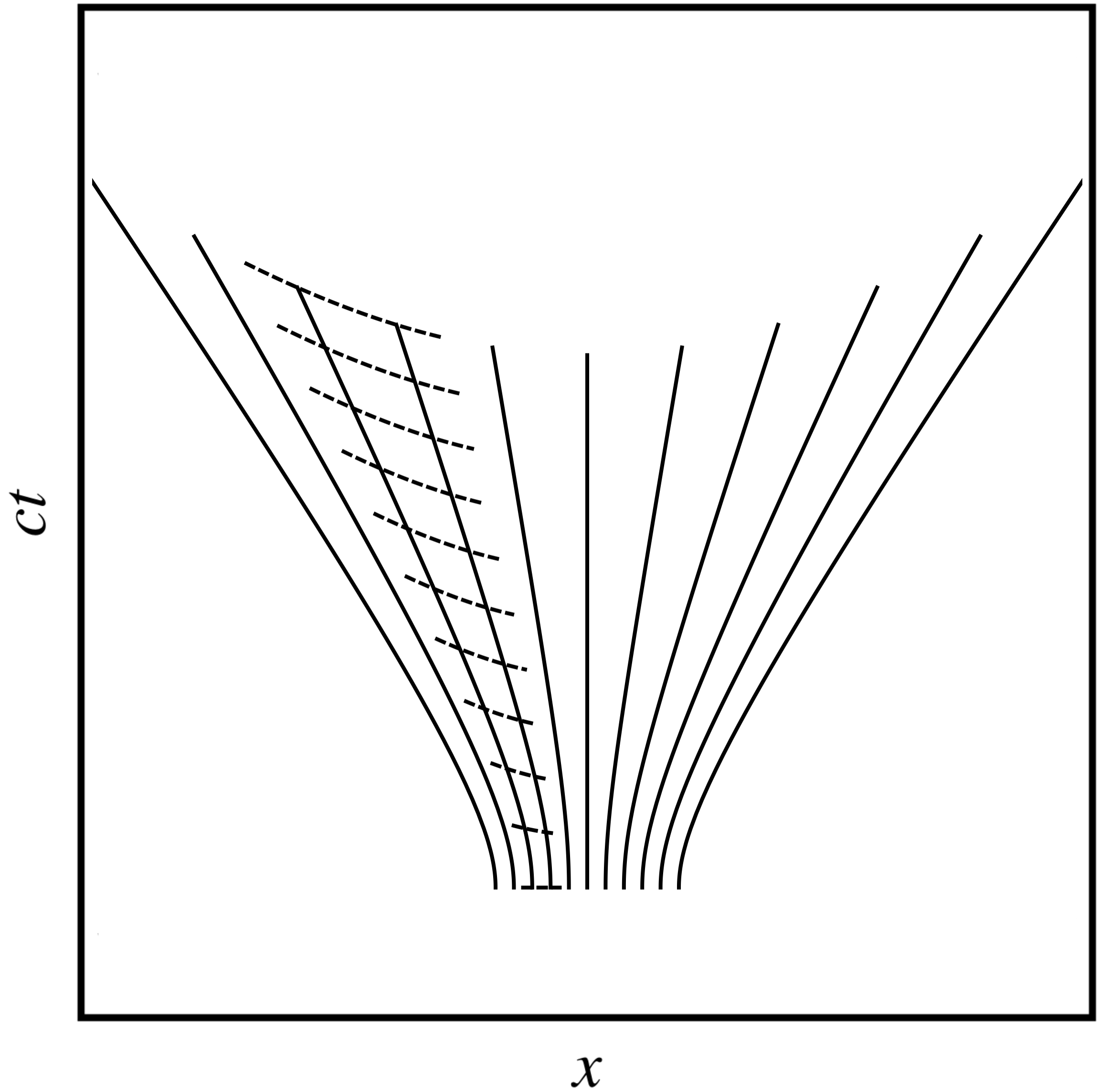

- The system now consists of an *ensemble* of quantum trajectories.
- Each trajectory has its own local simultaneity segments.
- Gluing all of these together, we can construct *global* simultaneity submanifolds.

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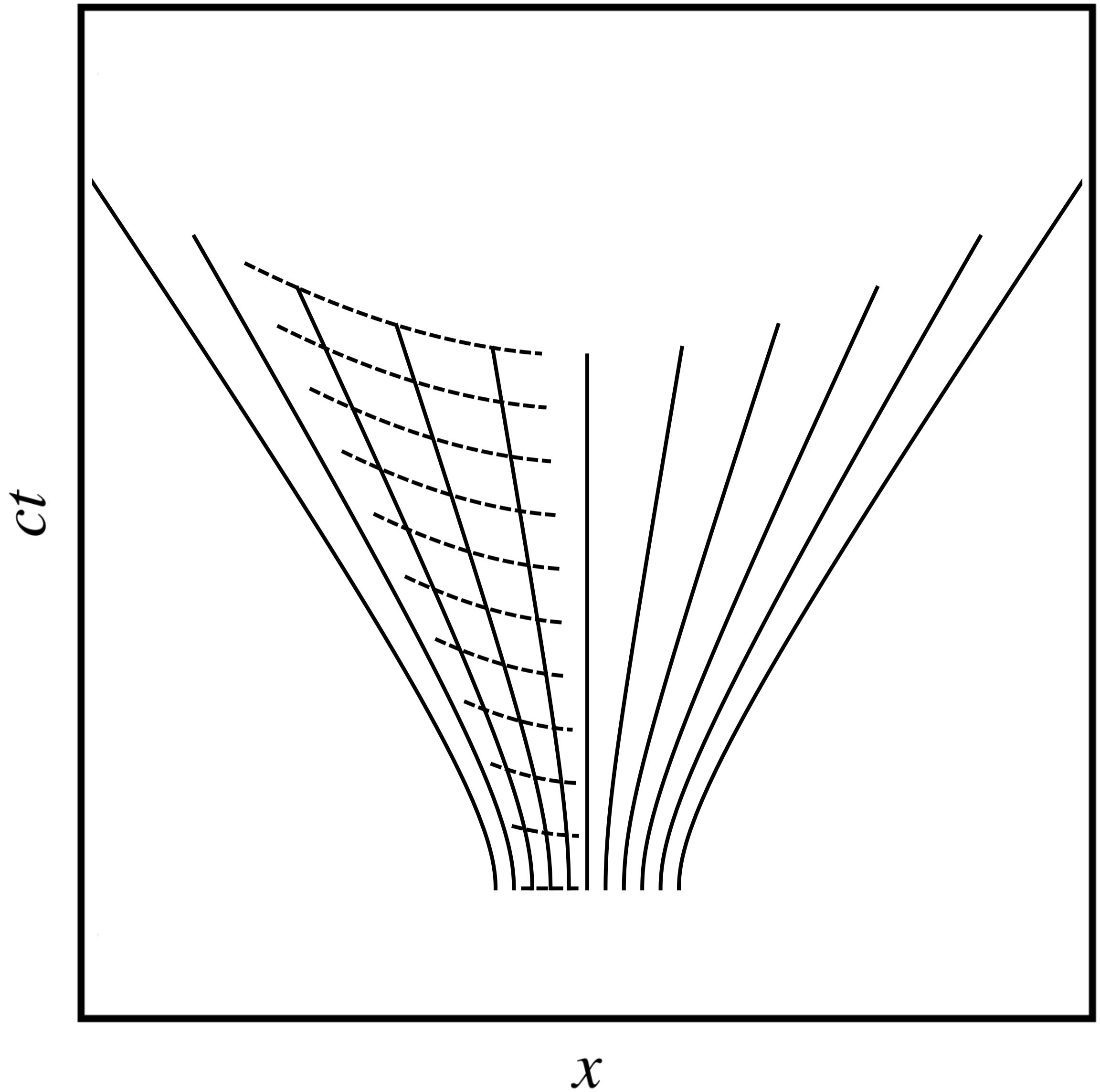

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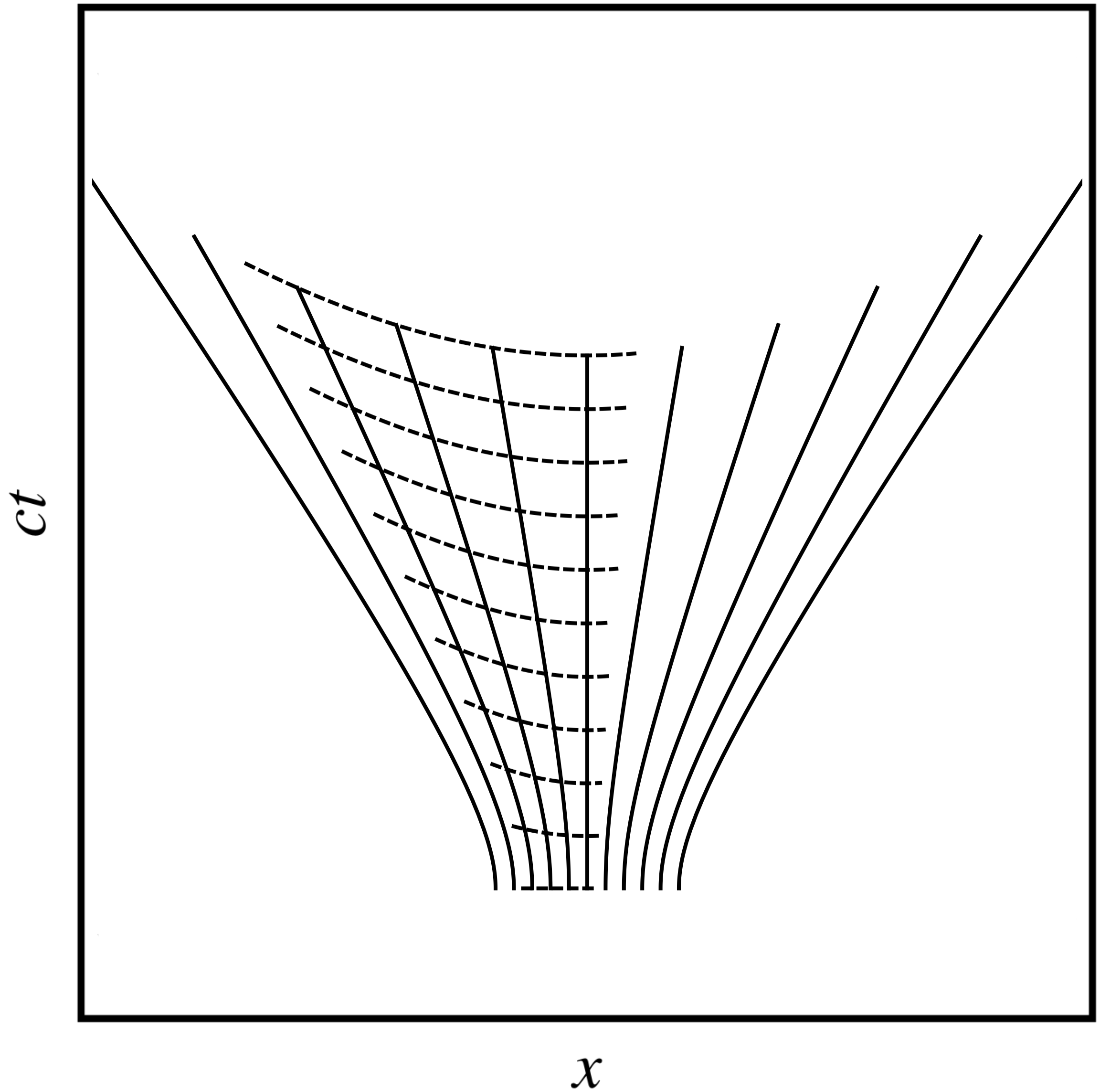

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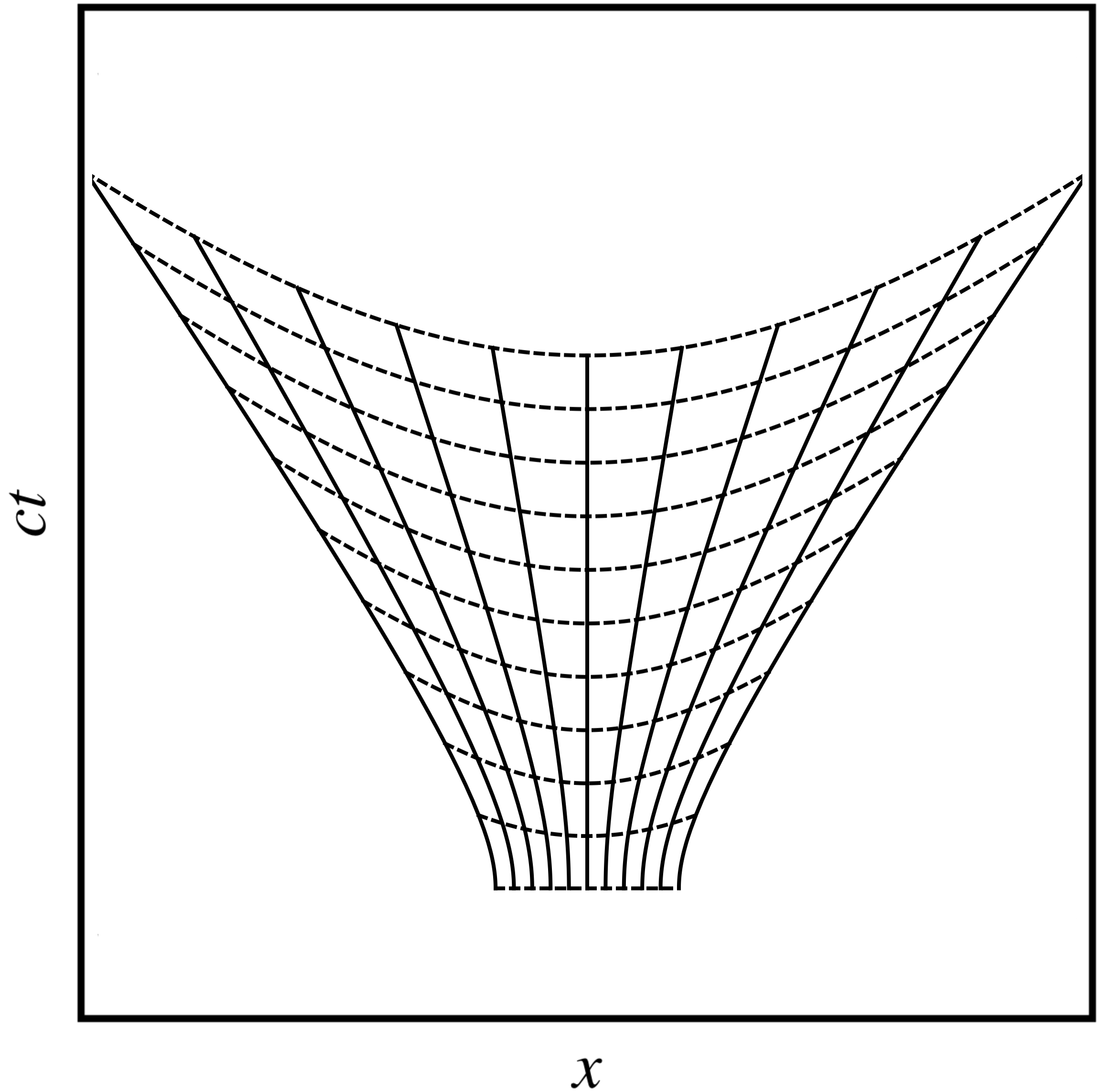

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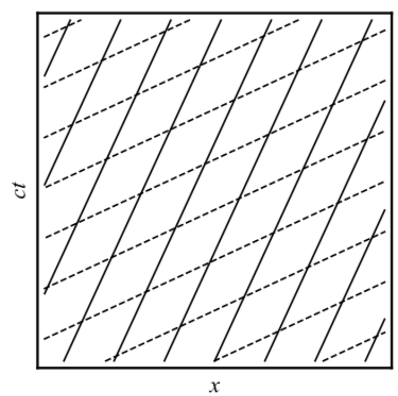
- Each trajectory has its own local simultaneity segments.
- Gluing all of these together, we can construct *global* simultaneity submanifolds.

Quantum Accelerated and Quantum Inertial Motion (1+1)

- Even for a single relativistic ***free*** particle, quantum forces can give rise to ***quantum accelerated motion***, i.e. curved quantum trajectories and simultaneity submanifolds.
- As a special case, a single relativistic free particle can also undergo ***quantum inertial motion***, when Q=0 everywhere.
  - Trajectories are parallel straight lines, corresponding to contours of Lorentz-transformed x'.
  - "Simultaneity submanifolds" are also parallel straight lines, corresponding to ct'



- This corresponds to the SR notion of an **inertial frame.**

Spacetime of a Relativistic Quantum Particle

- The *spacetime* of a single relativistic spin-zero particle is represented by a 4D Reimannian manifold, which is presumed *flat*.
- A global inertial frame can be defined. The inertial coordinates are: $x^\alpha = (ct, \mathbf{x})$ $\alpha = 0,1,2,3$
- Define the Minkowski metric tensor:

$$\tilde{\eta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The proper time is defined as:

$$\boxed{d\tau^2 = -\frac{1}{c^2}\,\eta_{\alpha\beta}\,dx^\alpha dx^\beta}$$

Ensemble Time and Natural Coords

- Simultaneity submanifolds are contours of a scalar function, called the *ensemble time* λ
- Define a system of *natural coordinates*:

$$X^{\mu} = \left(c\lambda, C^{i}\right)$$

where C^i are the trajectory labels. For now, we allow arbitrary reparametrizations: $\lambda \to \lambda' = \lambda'(\lambda)$ and $\mathbf{C} \to \mathbf{C}' = \mathbf{C}'(\mathbf{C})$

- The metric tensor of the natural coordinates and that of the inertial coordinates are related by:

$$\boxed{g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial X^{\mu}}\frac{\partial x^{\beta}}{\partial X^{\nu}}} \qquad \tilde{g} = \begin{pmatrix} g_{00} & \mathbf{0} \\ \mathbf{0} & \tilde{\gamma} \end{pmatrix}$$

- Note that the metric tensor is *block-diagonal*.
- Define:

$$g = \det\left(\tilde{g}\right) \qquad \gamma = \det\left(\tilde{\gamma}\right)$$

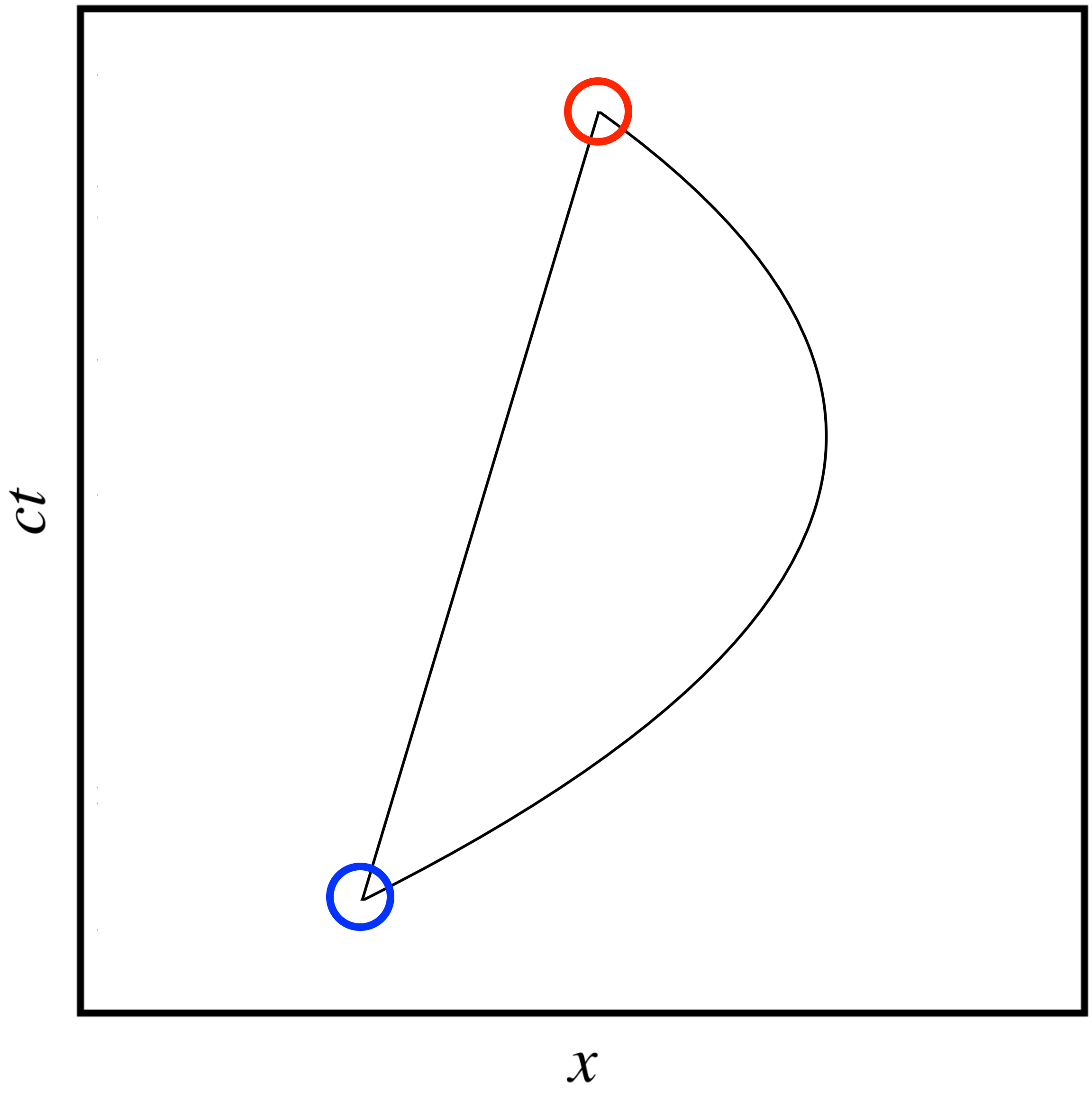
Ensemble Time and the Generalized Twin "Paradox"

- Note: the usual relativistic *proper time*, *τ,* can be defined along every trajectory in the ensemble, and is therefore a bona fide time *coordinate* on spacetime.
- *Is it possible to take τ to be an ensemble time, λ ?*
  - In general, *NO*, this is not possible.
  - The relation between τ and λ can be found from the metric tensor:

$$g_{00} = -\left(\frac{d\tau}{d\lambda}\right)^2$$

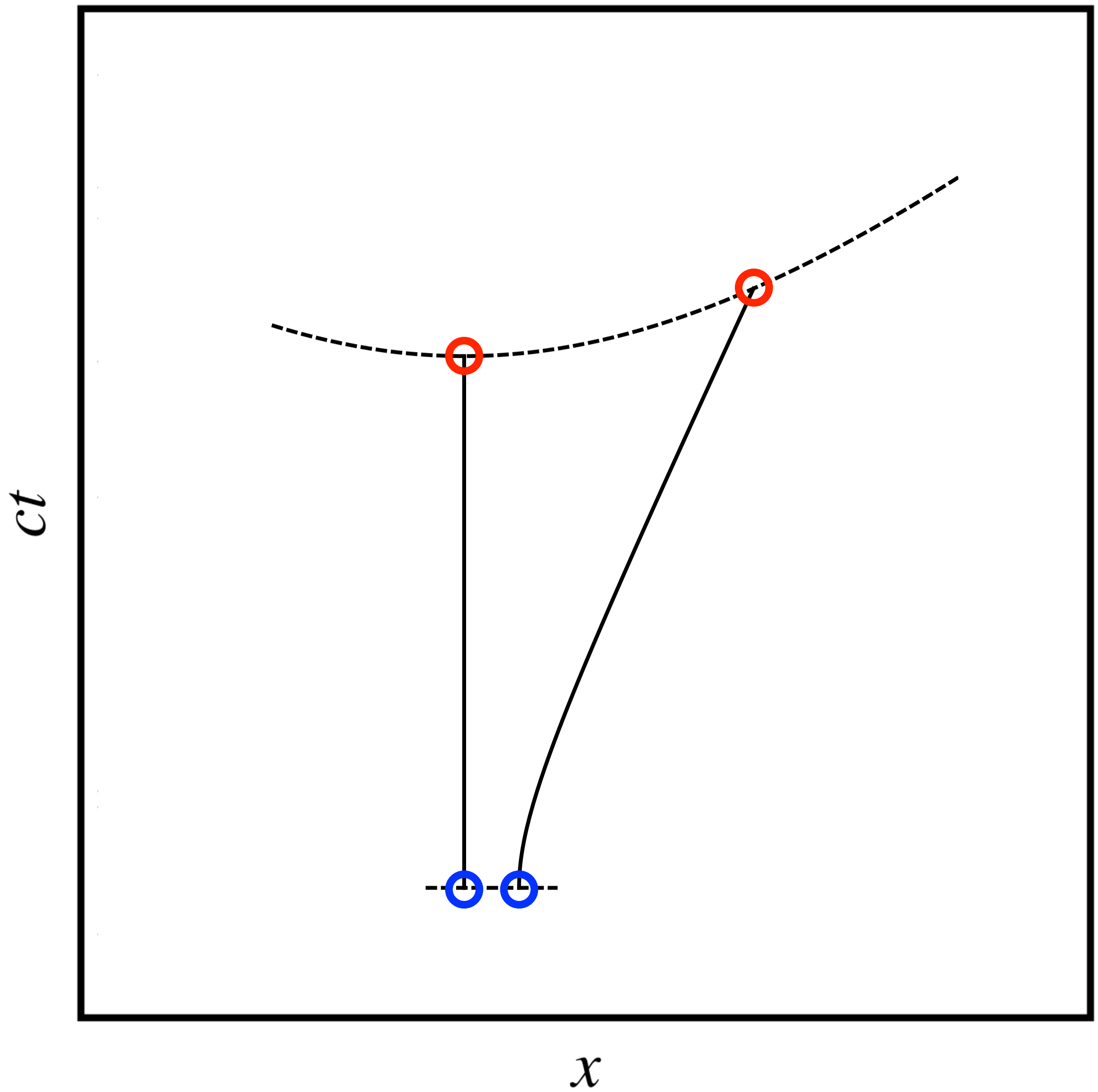
  - Note: g_{00} is *negative*, in keeping with the -+++ metric signature.
- The difference between τ and λ gives rise to the *generalized twin paradox.*

Regular Twin "Paradox"



- Two "twin" observers cross paths at the blue circle event.
- Left twin: inertial motion; right twin: accelerated motion
- Right twin is *younger* when paths recross at the red circle event.

Generalized (Quantum) Twin "Paradox"



- Two "copies" of the same observer follow two, non-crossing paths.
- Both agree that the two blue circle events occur simultaneously.
- Both also agree that the two red circle events occur simultaneously.
- One trajectory has experienced less elapsed proper time than the other.

Ensemble Proper Time, & the Relativistic Quantum Potential

- Of all choices of ensemble time coordinates, λ , one choice is special. We call it the *ensemble proper time*, denoted $\mathcal{T}$.
- There is a close connection between $\mathcal{T}$ and Q, the (relativistic) quantum potential:

$$\frac{d\tau}{d\mathcal{T}} = \tau_{\mathcal{T}} = \exp\left[-\frac{Q}{mc^2}\right]$$

- Note: $\mathcal{T}$ reduces to τ, in the limit of quantum inertial motion.
- *Note: Q itself plays a dynamical role, and not just its gradient, the quantum force!*
  - Reminiscent of the *gravitational potential*.

Ensemble Proper Time, & the Relativistic Quantum Potential

- Gravitational potential vs. quantum potential (weak-field limit)

$$\boxed{-\left(1+2\frac{m\Phi}{mc^2}\right) \approx g_{00} \approx -\left(1-2\frac{Q}{mc^2}\right)}$$

- Note: Q can be either positive *OR* negative!
- When $Q > 0$ (classically allowed), $d\tau < d\mathcal{T}$
  – The passage of the proper time for a given trajectory is **slower** than that of an inertial trajectory (time *dilation*).
- When $Q < 0$ (classically forbidden), $d\tau > d\mathcal{T}$
  – The passage of the proper time for a given trajectory is **faster** than that of an inertial trajectory (time *compression*).

Part II: Dynamics

[1] A. Bouda, Int. J. Mod. Phys. A 18, 3347 (2003).
[2] P. Holland, Ann. Phys. 315, 505 (2005).
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Non-relativistic Derivation

(1+1) Time-dependent Wavepackets

Spatial Probability Conservation:

- Equation (1) implies *conservation of spatial probability*:

$$\rho(x(t),t)dx(t) = \text{const} = f(C)dC$$

- Converting density from x to C space yields constant $f(C)$.
- *Uniformizing spatial coordinates*:
  - A convenient choice of spatial coordinate C is one for which the spatial density becomes uniform.
  - Specifically, for the special choice $C=P$, we require: $f(P) = 1$.
  - Leads to:

$$\rho = \left(1/x'\right)$$

Relativistic Derivation

Lagrangian and Action

- The action consists of two parts:

$$S = S_0 + S_Q \qquad S = \int d\lambda d^3 C f(\mathbf{C})\left(\mathcal{L}_0 - \mathcal{L}_Q\right)$$

$$\boxed{S_0 = \int d\lambda d^3 C\, f(\mathbf{C})\left(-mc^2\right)\sqrt{-\frac{1}{c^2}\eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial \lambda}\frac{\partial x^\alpha}{\partial \lambda}}}$$

- The Lagrangian is a homogeneous function of degree 1 in $\frac{\partial x^\alpha}{\partial \lambda}$. Thus the action is parameter invariant.
- We will set λ= $\mathcal{T}$ later on. We choose $\lambda = t$ to obtain the nonrelativistic limit.

$$\int (-mc^2)\sqrt{1 - \frac{\dot{x}^l}{c}\frac{\dot{x}_l}{c}}\, f(\mathbf{C})\, d^3C\, dt \approx \int \left(-mc^2 + \frac{1}{2}m\,\dot{x}^l\,\dot{x}_l\right) f(\mathbf{C})\, d^3C\, dt \qquad \ell = 1, 2, 3$$

Relativistic Derivation

Lagrangian and Action

- In *non-relativistic* quantum mechanics, it has been shown [J. Schiff and B. Poirier, (2012)] that the quantum potential is

$$Q[x^l_{C^i}, x^l_{C^iC^j}, x^l_{C^iC^jC^k}] = -\frac{\hbar^2}{2m}\left(\frac{1}{\gamma^{1/4} f^{1/2}}\right) \times \partial_i\left[\gamma^{1/2}\gamma^{ij}\,\partial_j\left(\frac{f^{1/2}}{\gamma^{1/4}}\right)\right] \qquad \ell = 1,2,3$$

- The following *relativistic* quantum potential is proposed:

$$Q[x^\alpha_{C^i}, x^\alpha_{C^iC^j}, x^\alpha_{C^iC^jC^k}] = -\frac{\hbar^2}{2m}\left(\frac{1}{\gamma^{1/4} f^{1/2}}\right) \times \partial_i\left[\gamma^{1/2}\gamma^{ij}\,\partial_j\left(\frac{f^{1/2}}{\gamma^{1/4}}\right)\right] \qquad \alpha = 0,1,2,3$$

- This expression suggests that the quantum part of the action is:

$$\boxed{S_Q = \int d\lambda d^3C\, f(\mathbf{C})\left[-\frac{\hbar^2}{8m}\frac{d\tau}{d\lambda}\left(\frac{f(\mathbf{C})}{\sqrt{\gamma}}\right)^{-2}\gamma^{ij}\frac{\partial}{\partial C^i}\left(\frac{f(\mathbf{C})}{\sqrt{\gamma}}\right)\frac{\partial}{\partial C^j}\left(\frac{f(\mathbf{C})}{\sqrt{\gamma}}\right)\right]}$$

Relativistic Derivation

Dynamical PDE (eqs. of motion)

- By extremizing the action, we obtain the equation of motion for the trajectory ensemble.

$$\boxed{\frac{\partial^2 x^\alpha}{\partial \mathcal{T}^2} = \exp\left[-\frac{2Q}{mc^2}\right]\frac{f^\alpha}{m} - \frac{1}{mc^2}\frac{\partial Q}{\partial \mathcal{T}}\frac{\partial x^\alpha}{\partial \mathcal{T}}}$$

- PDE is fourth order in **C**, second order in $\mathcal{T}$, *but* treats all inertial coordinates x^α on equal footing.
- Choosing uniformizing coordinates: $f(\mathbf{C}=\mathbf{P})=1$

$$Q = -\frac{\hbar^2}{2m}\gamma^{-1/4}\frac{\partial}{\partial C^i}\left[\gamma^{1/2}\gamma^{ij}\frac{\partial}{\partial C^j}\gamma^{-1/4}\right], \qquad f^\alpha = -\frac{\partial x^\alpha}{\partial C^i}\gamma^{ij}\frac{\partial}{\partial C^j}Q$$

Trajectories vs. Klein-Gordon

	Klein-Gordon equation	**Relativistic quantum trajectories**
Dependence on future time	Dynamics depends on the past and future time. (Violates "causality.")	Quantum force does ***not*** depend on the past and future time. (Satisfying "causality.")
Temporal part of 4-current density	***not*** positive-definite in general.	is **positive-definite** in general.
4-current density	4-current density ***not*** time-like in general.	4-current density is **time-like** in general
order of the PDE	2^{nd} order in both space and time.	2^{nd} order in time ($\mathcal{T}$), 4^{th} order in space (**C**)

- The relativistic PDE is *equivalent* to the Klein-Gordon equation, except for the fact that the quantum force vector has no ensemble time ($\mathcal{T}$) component (it "lives" on the simultaneity submanifold).

Conservation Laws

- From Noether's theorem, we obtain the Euler-Lagrange equation

$$\boxed{\frac{\partial \mathcal{L}}{\partial x^\alpha} - \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda x^\alpha)} \right) - \partial_C \left(\frac{\partial \mathcal{L}}{\partial(\partial_C x^\alpha)} \right) + \partial_C^2 \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_C^2 x^\alpha\right)} \right) = 0}$$

- as well as the Noether current equation

$$\boxed{\partial_\mu \Theta^\mu{}_\sigma = 0}$$

$$x^\alpha = x^\alpha(X)$$

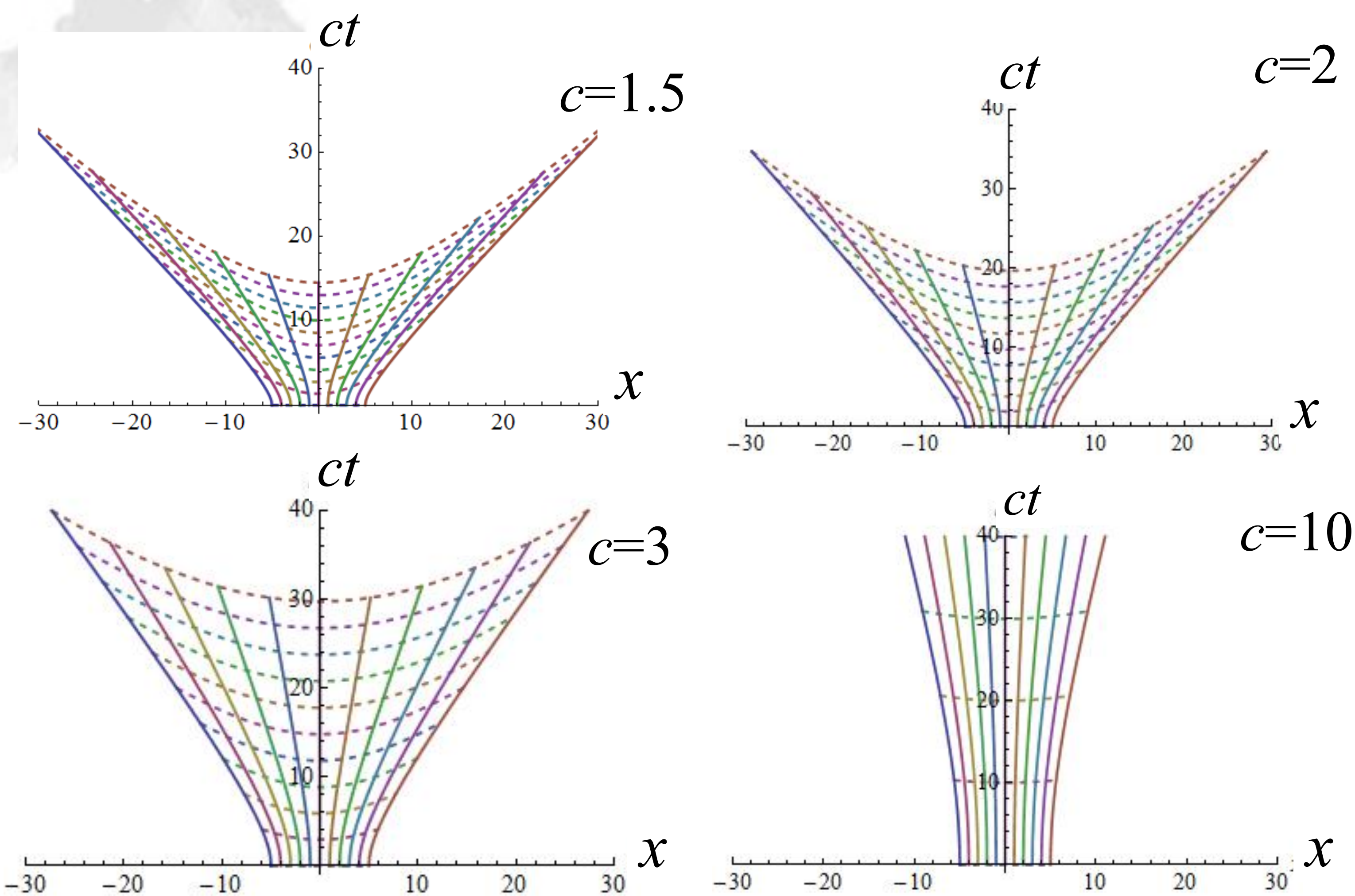
$$\partial_\mu \equiv \frac{\partial}{\partial X^\mu}$$

where

$$\boxed{\begin{aligned}\Theta^\mu{}_\sigma = &\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu x^\alpha)} - \sum_{\rho,\nu} \frac{1}{2} \delta^{\mu\rho} \left(1 + \delta_{\rho\nu}\right) \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \right] \partial_\sigma x^\alpha \\ &+ \sum_{\rho,\nu} \frac{1}{2} \delta^{\mu\rho} \left(1 + \delta_{\rho\nu}\right) \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu x^\alpha)} \partial_\nu \partial_\sigma x^\alpha - \delta^\mu{}_\sigma \mathcal{L}\end{aligned}}$$

- The conservation of energy and momentum can be obtained from these equations.

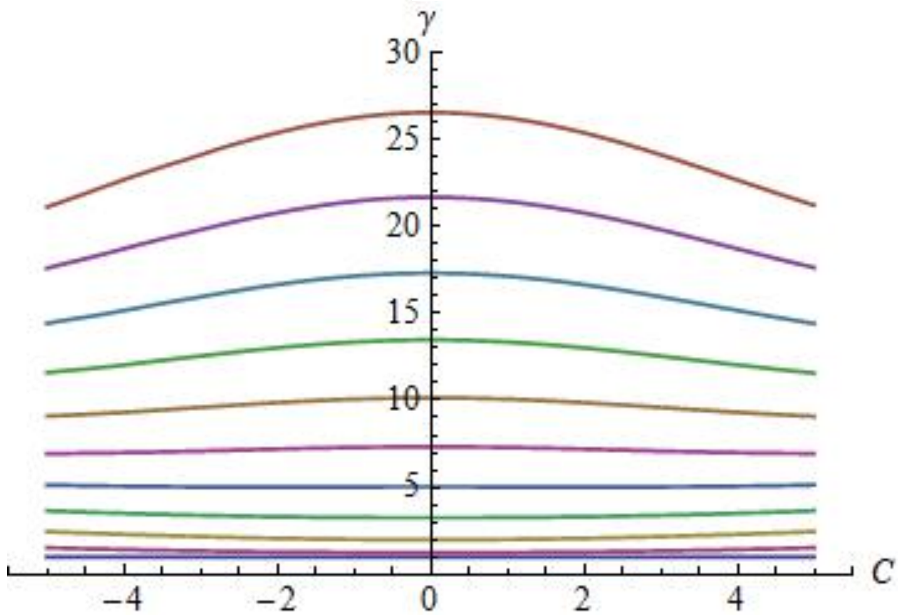
Gaussian Wavepacket



Gaussian Wavepacket

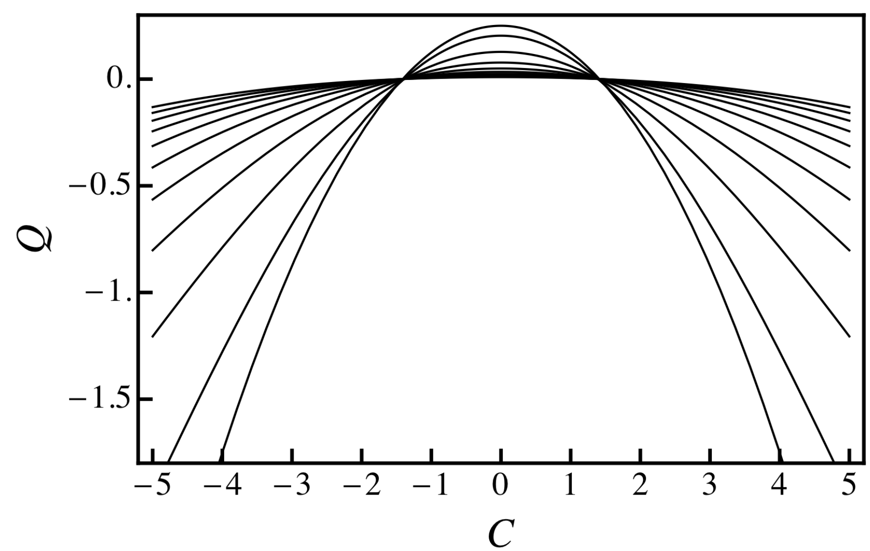
c=3

γ



C

Figure 5



Acknowledgments:

Personnel:

- Jeremy Schiff:
  - Bar-Ilan University, Ramat Gan, Israel
- Gerard Parlant:
  - CNRS, Montpellier II University, France
- Postdoctoral Researchers:
  - Hung-Ming Tsai
  - Yong-Cheng Ou
  - Kisam Park
- Graduate Students:
  - Chaowen Guo

Funding:

- Robert A. Welch Foundation
- National Science Foundation

  Small Grant for Exploratory Research

Conference Organizers:

Gerhard Grössing

Jan Walleczek