Cooperative Dynamical Processes:
Emergence of Relativistic Quantum Theory

Petr Jizba[1,2]

[1] ITP, Freie Universität Berlin

[2] FNSPI, Czech Technical University in Prague

in collaboration with F. Scardigli  and H. Kleinert 

EmQM13, Vienna, 4th October, 2013

Prologue

" Behind the apparent Lorentz invariance of the phenomena, there is a deeper level which is not Lorentz invariant ... "

- Interview in Davis and Brown's *The Ghost in the Atom*

John Bell 1986

Outline

1. *From QM to stochastic processes and back*
   - Path-integral appetizer
2. *Superstatistics paradigm*
   - Old wine in new bottles
   - Chapman–Kolmogorov eq. for Markov process
3. *Applications to relativistic QM*
   - Klein–Gordon particle
   - Emergent relativity
   - Gravity

Outline

Feynman's sum over histories

Let us have a particle (elementary excitation) at a spacetime point x_A^μ

Probability $\mathcal{P}$ of finding it at another point x_B^μ is given according to QM by

$$\mathcal{P}(x_A, x_B) \ = \ |\langle x_B | x_A \rangle|^2$$

$\langle x_B | x_A \rangle$ is computed as a sum over all possible histories connecting x_A and x_B.

$$\langle x_B | x_A \rangle \ = \ \sum_{P_{x_A x_B}} \exp(i/\hbar \, S[P_{x_A x_B}]) \ = \ \int_{x_A}^{x_B} \mathcal{D}x \ \exp(i/\hbar \, S[x])$$

$P_{x_A x_B}$ is a particular history, S is the action.

Feynman's sum over histories

Let us have a particle (elementary excitation) at a spacetime point x_A^μ

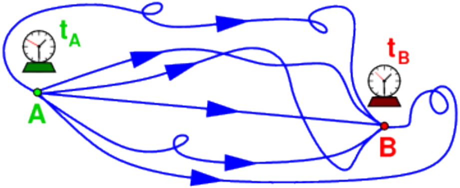
Probability $\mathcal{P}$ of finding it at another point x_B^μ is given according to QM by

$$\mathcal{P}(x_A, x_B) = |\langle x_B|x_A\rangle|^2$$

$\langle x_B|x_A\rangle$ is computed as a sum over all possible histories connecting x_A and x_B.

$$\langle x_B|x_A\rangle = \sum_{P_{x_A x_B}} \exp(i/\hbar\, S[P_{x_A x_B}]) = \int_{x_A}^{x_B} \mathcal{D}x\, \exp(i/\hbar\, S[x])$$

$P_{x_A x_B}$ is a particular history, S is the action.



Feynman's sum over histories

Let us have a particle (elementary excitation) at a spacetime point x_A^{μ}

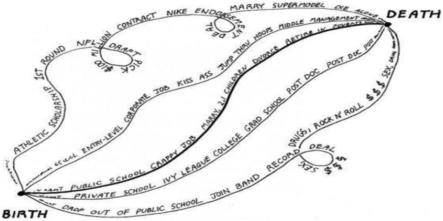
Probability $\mathcal{P}$ of finding it at another point x_B^{μ} is given according to QM by

$$\mathcal{P}(x_A, x_B) \;=\; |\langle x_B | x_A \rangle|^2$$

$\langle x_B | x_A \rangle$ is computed as a sum over all possible histories connecting x_A and x_B.

$$\langle x_B | x_A \rangle \;=\; \sum_{P_{x_A x_B}} \exp(i/\hbar\, S[P_{x_A x_B}]) \;=\; \int_{x_A}^{x_B} \mathcal{D}x \;\exp(i/\hbar\, S[x])$$

$P_{x_A x_B}$ is a particular history, S is the action.



Kac's sum over random walks

PI also relates *quantum* and *stochastic processes*.

This is due to formal analogy between *Schrödinger* and *diffusion* eq.

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi = \hat{H}\psi \quad \leftrightarrow \quad \frac{\partial P}{\partial t} = D\Delta P$$

$\Rightarrow$ *QM* goes over to *Brownian motion* when

$$it \mapsto t \;\; and \;\; \hbar^2/2m \mapsto D$$

So analytic continuation of QM PI gives PI where one sums over all possible random walks (Wiener integral).

$$\langle x_B, t_B | x_A, t_A \rangle = \int_{x(t_A)=x}^{x(t_B)=x'} \mathcal{D}x \; \exp\left[\frac{i}{\hbar}\int_{t_A}^{t_B} dt\, \frac{m\dot{x}^2}{2}\right] \mapsto$$

$$P(x_B, t_B | x_A, t_A) = \int_{x(t_A)=x}^{x(t_B)=x'} \mathcal{D}x \; \exp\left[-\frac{1}{4D}\int_{t_A}^{t_B} dt\, \dot{x}^2\right]$$

Outline

What is so *Super* about Superstatistics?

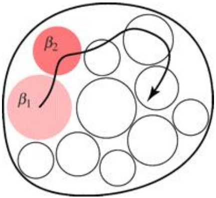
Superstatistics

C. Beck

(1959 - $*$)

Complex systems often exhibit a dynamics that can be regarded as a superposition of several dynamics on different time scales. ... The two effects produce a superposition of two statistics, or in a short, a *"superstatistics"*.*

* C. Beck et al., Physica **A** 322 (2003); C. Beck, Phys. Rev. Lett. **98** (2007)



Movement through different temperature zones.

Probability to find a brownian particle to have velocity v is

$$p(\mathbf{v}) = \int e^{-\beta m \mathbf{v}^2/2} f(\beta) d\beta$$

What is so *Super* about Superstatistics?

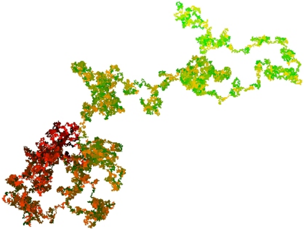
Superstatistics

C. Beck

(1959 - $*$)

Complex systems often exhibit a dynamics that can be regarded as a superposition of several dynamics on different time scales. ... The two effects produce a superposition of two statistics, or in a short, a *"superstatistics"*.*

* C. Beck et al., Physica **A** 322 (2003); C. Beck, Phys. Rev. Lett. **98** (2007)



Brownian particle in inhomogeneous fluid environment.

Probability to find a brownian particle to have velocity v is

$$p(\mathbf{v}) = \int e^{-\beta m \mathbf{v}^2/2} f(\beta) d\beta$$

Outline

Superstatistics and PI

Key property of PI in *statistical* physics is that conditional PDF's fulfill

Chapman-Kolmogorov equation for continuous Markovian processes

S. Chapman
(1888-1970)

$$P(x_b, t_b|x_a, t_a) = \int_{-\infty}^{\infty} dx\, P(x_b, t_b|x, t)P(x, t|x_a, t_a)$$

A.N. Kolmogorov
(1903-1987)

Conversely, any probability satisfying C-K-E possesses PI representation !!

One often encounters probabilities formulated as a superposition of PI

$$\bar{P}(x_b, t_b|x_a, t_a) \;=\; \int_0^{\infty} dv\, \omega(v, t_{ba}) \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x\mathcal{D}p\; e^{\int_{t_a}^{t_b} d\tau (ip\dot{x} - vH(p,x))}$$

$\omega(v, t_{ba})$ with ($t_{ba} = t_b - t_a$) cont. and norm. PDF on $\mathbb{R}^+ \times \mathbb{R}^+$.

Superstatistics and PI

Q: Is it possible that also $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ satisfies C-K-E?

A: Yes, if $\omega(v, t)$ fulfills a certain simple functional equation*:

- Define a rescaled weight function

$$w(v,t) \equiv \omega(v/t, t)/t$$

- Calculate its Laplace transform

$$\tilde{w}(p_v, t) \equiv \int_0^\infty dv\, e^{-p_v v} w(v,t)$$

- $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ satisfies C-K-E iff

$$\tilde{w}(p_v, t_1 + t_2) = \tilde{w}(p_v, t_2)\tilde{w}(p_v, t_1)$$

*P.J. and H. Kleinert, PRE 78 (2008) 031122, B. Simons, J. Funct. An. 91 (1990) 117

Superstatistics and PI

Q: Is it possible that also $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ satisfies C-K-E?

A: Yes, if $\omega(v, t)$ fulfills a certain simple functional equation*:

- Define a rescaled weight function

$$w(v, t) \equiv \omega(v/t, t)/t$$

- Calculate its Laplace transform

$$\tilde{w}(p_v, t) \equiv \int_0^\infty dv\, e^{-p_v v} w(v, t)$$

- $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ satisfies C-K-E iff

$$\tilde{w}(p_v, t_1 + t_2) = \tilde{w}(p_v, t_2)\tilde{w}(p_v, t_1)$$

*P.J. and H. Kleinert, PRE **78** (2008) 031122, B. Simons, J. Funct. An. **91** (1990) 117

Superstatistics and PI

Q: Is it possible that also $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ satisfies C-K-E?

A: Yes, if $\omega(v, t)$ fulfills a certain simple functional equation*:

- Define a rescaled weight function

$$w(v,t) \;\equiv\; \omega(v/t, t)/t$$

- Calculate its Laplace transform

$$\tilde{w}(p_v, t) \;\equiv\; \int_0^\infty \mathrm{d}v\, e^{-p_v v}\, w(v,t)$$

- $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ satisfies C-K-E iff

$$\tilde{w}(p_v, t_1 + t_2) \;=\; \tilde{w}(p_v, t_2)\tilde{w}(p_v, t_1)$$

* P.J. and H. Kleinert, PRE **78** (2008) 031122, B. Simons, J. Funct. An. **91** (1990) 117

Superstatistics and PI

Q: Is it possible that also $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ satisfies C-K-E?

A: Yes, if $\omega(v, t)$ fulfills a certain simple functional equation*:

- Define a rescaled weight function

$$w(v, t) \equiv \omega(v/t, t)/t$$

- Calculate its Laplace transform

$$\tilde{w}(p_v, t) \equiv \int_0^\infty \mathrm{d}v\, e^{-p_v v} w(v, t)$$

- $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ satisfies C-K-E iff

$$\tilde{w}(p_v, t_1 + t_2) = \tilde{w}(p_v, t_2)\tilde{w}(p_v, t_1)$$

* P.J. and H. Kleinert, PRE **78** (2008) 031122, B. Simons, J. Funct. An. **91** (1990) 117

Superstatistics and PI

Q: Is it possible that also $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ satisfies C-K-E?

A: Yes, if $\omega(v, t)$ fulfills a certain simple functional equation*:

- Define a rescaled weight function

$$w(v, t) \equiv \omega(v/t, t)/t$$

- Calculate its Laplace transform

$$\tilde{w}(p_v, t) \equiv \int_0^\infty \mathrm{d}v\, e^{-p_v v} w(v, t)$$

- $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ satisfies C-K-E iff

$$\tilde{w}(p_v, t_1 + t_2) = \tilde{w}(p_v, t_2)\tilde{w}(p_v, t_1)$$

* P.J. and H. Kleinert, PRE **78** (2008) 031122, B. Simons, J. Funct. An. **91** (1990) 117

Superstatistics and PI

- Assuming continuity in t,$w(p_v, t)$ is unique and can be written as:

$$\tilde{w}(p_v, t) \;=\; [G(p_v)]^t \;=\; e^{-tF(p_v)}$$

$F(p_v)$ satisfies the *Lévy–Khintchine formula*

$\Rightarrow$ $w(v, t)$ is *infinitely divisible distribution*

- Laplace inverse of $\tilde{w}(p_v, t)$ yields $\omega(v, t)$.

"Emergent" Hamiltonian *

Once above conditions are satisfied, then $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ possesses a *path integral representation on its own*. New Hamiltonian is given by

$$\overline{H}(\boldsymbol{p}, \boldsymbol{x}) \;=\; F(H(\boldsymbol{p}, \boldsymbol{x}))$$

* P.J. and H. Kleinert, PRE **78** (2008) 031122

Superstatistics and PI

- Assuming continuity in t,$w(p_v, t)$ is unique and can be written as:

$$\tilde{w}(p_v, t) = [G(p_v)]^t = e^{-tF(p_v)}$$

$F(p_v)$ satisfies the *Lévy–Khintchine formula*

$\Rightarrow w(v, t)$ is *infinitely divisible distribution*

- Laplace inverse of $\tilde{w}(p_v, t)$ yields $\omega(v, t)$.

"Emergent" Hamiltonian *

Once above conditions are satisfied, then $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ possesses a *path integral representation on its own*. New Hamiltonian is given by

$$\overline{H}(\boldsymbol{p}, \boldsymbol{x}) = F(H(\boldsymbol{p}, \boldsymbol{x}))$$

* P.J. and H. Kleinert, PRE **78** (2008) 031122

Superstatistics and PI

- Assuming continuity in t,$w(p_v, t)$ is unique and can be written as:

$$\tilde{w}(p_v, t) \ = \ [G(p_v)]^t \ = \ e^{-tF(p_v)}$$

$F(p_v)$ satisfies the *Lévy–Khintchine formula*

$\Rightarrow$ $w(v, t)$ is *infinitely divisible distribution*

- Laplace inverse of $\tilde{w}(p_v, t)$ yields $\omega(v, t)$.

"Emergent" Hamiltonian *

Once above conditions are satisfied, then $\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ possesses a *path integral representation on its own*. New Hamiltonian is given by

$$\overline{H}(\boldsymbol{p}, \boldsymbol{x}) \ = \ F(H(\boldsymbol{p}, \boldsymbol{x}))$$

* P.J. and H. Kleinert, PRE **78** (2008) 031122

Outline

Some observations

Consider $G(x) \;=\; e^{-a\sqrt{x}}$ with $a \in \mathbb{R}^+$ this gives $\omega(v, a, t) \;=\; \dfrac{a \exp\left(-a^2 t/4v\right)}{2\sqrt{\pi}\sqrt{v^3/t}}$

$\omega(v, t)$ is the *Weibull distribution* of order a. For W.D. of order 1 we have *

$$P(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a) \;=\; \int_{\boldsymbol{x}(t_a)=\boldsymbol{x}_a}^{\boldsymbol{x}(t_b)=\boldsymbol{x}_b} \mathcal{D}\boldsymbol{x}\, \frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \exp\left\{\int_{t_a}^{t_b} \mathrm{d}\tau \left[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - c\sqrt{\boldsymbol{p}^2 + m^2c^2}\right]\right\}$$

$$= \int_0^\infty \mathrm{d}v\, \omega(v, t_{ba}) \int_{\boldsymbol{x}(t_a)=\boldsymbol{x}_a}^{\boldsymbol{x}(t_b)=\boldsymbol{x}_b} \mathcal{D}\boldsymbol{x}\, \frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \exp\left\{\int_{t_a}^{t_b} \mathrm{d}\tau \left[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - v(\boldsymbol{p}^2c^2 + m^2c^4)\right]\right\}$$

Upper eq. is propagator for *relativistic scalar particle*, lower can be considered as a superposition of *non-relativistic* free-particle PI

* P.J. and H. Kleinert, PRE **78** (2008) 031122; PRD **82** (2010) 085016; arXiv:1007.3922

Outline

Fluctuating mass

The *relativistic PI* can be rewritten in yet another way, namely

$$\int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}''} \mathcal{D}\boldsymbol{x} \frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \exp\left\{ \int_0^t \mathrm{d}\tau \left[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} \ - \ c\sqrt{\boldsymbol{p}^2 + m^2c^2} \right] \right\}$$

$$= \int_0^\infty \mathrm{d}\tilde{m}\, f_{\frac{1}{2}}\left(\tilde{m}, tc^2, tc^2m^2\right) \int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}''} \mathcal{D}\boldsymbol{x} \frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \exp\left\{ \int_0^t \mathrm{d}\tau \left[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - \frac{\boldsymbol{p}^2}{2\tilde{m}} - mc^2 \right] \right\}$$

where

$$f_p(z,a,b) \ = \ \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})}\, z^{p-1}\, \mathrm{e}^{-(az+b/z)/2}$$

is the *generalized inverse Gaussian* distribution.

Fluctuating mass

So $\tilde{m}$ plays role of *Newtonian mass* which takes on continuous values distributed according to $f_{\frac{1}{2}}\left(\tilde{m}, tc^2, tc^2 m^2\right)$ with $\langle \tilde{m} \rangle = m + 1/tc^2$.

Heuristic interpretation:

Single-particle relativistic th. can be viewed as a single-particle non-relativistic th. whose Newtonian mass $\tilde{m}$ represents a fluctuation par. approaching on average the Einsteinian rest mass m in the large t limit.

On more speculative vein ...

We can fit the above observation into currently much debated *emergent (special) relativity*. ER views relativity as a theory that statistically emerges from a deeper (essentially non-relativistic) level of dynamics.

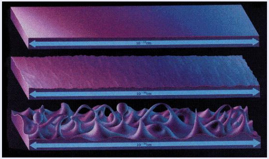
Emergent special relativity

In order to reconcile GR and QM a dramatic conceptual shift is required in our understanding of a *spacetime** $\Rightarrow$

Revival of the idea of spacetime as a discrete coarse-grained structure (typically at Planckian lengths $\ell_{\rm p} \approx 10^{-35}$m) $\Rightarrow$

Quantum-gravity models:

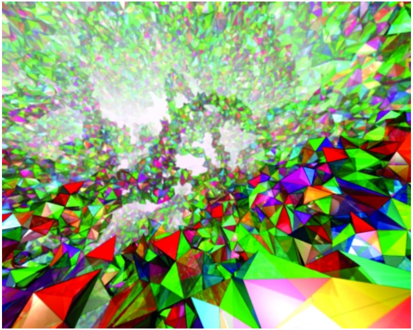
- space-time foam (Wheeler - 1955)
- loop quantum gravity
- causal dynamical triangulation
- black-hole physics
- cosmic cellular automata (Wolfram - 04)



space-time foam

* D. Bohm, 1954; G. 't Hooft, J. Stat. Phys. **53** (1988) 323
P.J., H. Kleinert and F. Scardigli, PRD **81** (2010) 084030

World crystal



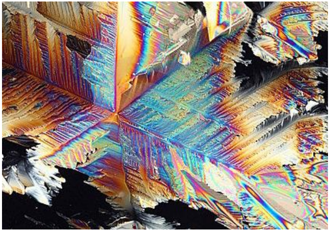
Vision of the spacetime in loop quantum gravity.
M. Bojowald, PRL 86 (2001) 5227

Fluctuations of the Newtonian mass can be understood as originating from particle's evolution in *"granular" medium* — "spacetime".

World polycrystalline

Granularity typically leads to corrections in the *local dispersion relation* and hence to modifications in *local effective mass*.

World-polycrystalline paradigm:



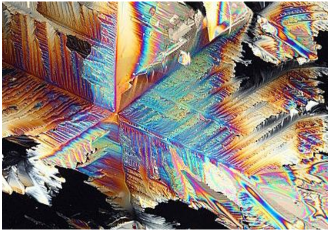
polycrystalline of copper

polycrystalline of silicon

World polycrystalline

Granularity typically leads to corrections in the *local dispersion relation* and hence to modifications in *local effective mass*.

World-polycrystalline paradigm:



polycrystalline of copper

polycrystalline of silicon

World polycrystalline

Granularity typically leads to corrections in the *local dispersion relation* and hence to modifications in *local effective mass*.

World-polycrystalline paradigm:

- On the fast-time level a non-relativistic particle propagates through grains with a different local $\tilde{m}$ in each grain.
- On the long-time scale the probability of the distribution of $\tilde{m}$ in various grains is $f_{\frac{1}{2}}\left(\tilde{m}, tc^2, tc^2 m^2\right)$.
- Because the fast-time scale motion is *brownian*, local PDM conditioned on some fixed $\tilde{m}$ in a given grain is *Gaussian*:

$$\hat{\rho}(\boldsymbol{p}, t|\tilde{m}) \propto e^{-t\hat{\boldsymbol{p}}^2/2\tilde{m}}$$

World polycrystalline

Granularity typically leads to corrections in the *local dispersion relation* and hence to modifications in *local effective mass*.

World-polycrystalline paradigm:

- On the fast-time level a non-relativistic particle propagates through grains with a different local $\tilde{m}$ in each grain.
- On the long-time scale the probability of the distribution of $\tilde{m}$ in various grains is $f_{\frac{1}{2}}\left(\tilde{m}, tc^2, tc^2 m^2\right)$.
- Because the fast-time scale motion is *brownian*, local PDM conditioned on some fixed $\tilde{m}$ in a given grain is *Gaussian*:

$$\hat{\rho}(\boldsymbol{p}, t|\tilde{m}) \propto e^{-t\hat{\boldsymbol{p}}^2/2\tilde{m}}$$

World polycrystalline

Granularity typically leads to corrections in the *local dispersion relation* and hence to modifications in *local effective mass*.

World-polycrystalline paradigm:

- On the fast-time level a non-relativistic particle propagates through grains with a different local $\tilde{m}$ in each grain.
- On the long-time scale the probability of the distribution of $\tilde{m}$ in various grains is $f_{\frac{1}{2}}\left(\tilde{m}, tc^2, tc^2 m^2\right)$.
- Because the fast-time scale motion is *brownian*, local PDM conditioned on some fixed $\tilde{m}$ in a given grain is *Gaussian*:

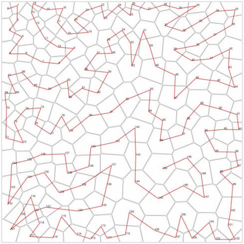
$$\hat{\rho}(\boldsymbol{p}, t|\tilde{m}) \ \propto \ e^{-t\hat{\boldsymbol{p}}^2/2\tilde{m}}$$

World polycrystalline

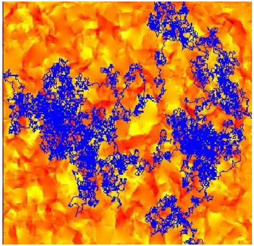
- *Joint PDM* is $\hat{\rho}(\boldsymbol{p}, t; \tilde{m}) = f_{\frac{1}{2}}\left(\tilde{m}, tc^2, tc^2 m^2\right) \hat{\rho}(\boldsymbol{p}, t|\tilde{m})$
- *marginal PDM* describing the mass-averaged (i.e. long-time) behavior is

$$\hat{\rho}(\boldsymbol{p}, t) = \int_0^\infty \mathrm{d}\tilde{m}\, f_{\frac{1}{2}}\left(\tilde{m}, tc^2, tc^2 m^2\right) \hat{\rho}(\boldsymbol{p}, t|\tilde{m})$$

Evolution in discrete spacetime:



Typical trajectory in Voronoi geometry



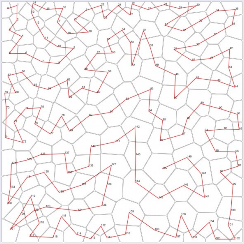
Polycrystalline universe

World polycrystalline

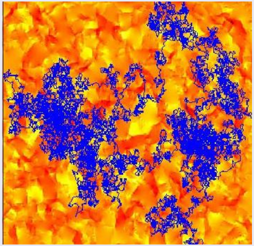
- *Joint PDM* is $\hat{\rho}(\boldsymbol{p}, t; \tilde{m}) = f_{\frac{1}{2}}\left(\tilde{m}, tc^2, tc^2 m^2\right) \hat{\rho}(\boldsymbol{p}, t|\tilde{m})$
- *marginal PDM* describing the mass-averaged (i.e. long-time) behavior is

$$\hat{\rho}(\boldsymbol{p}, t) = \int_0^\infty \mathrm{d}\tilde{m}\, f_{\frac{1}{2}}\left(\tilde{m}, tc^2, tc^2 m^2\right) \hat{\rho}(\boldsymbol{p}, t|\tilde{m})$$

Evolution in discrete spacetime:



Typical trajectory in Voronoi geometry



Polycrystalline universe

Important scales:

Some simple consequences:

- $\langle \tilde{m} \rangle = m + 1/tc^2$ and $\text{var}(\tilde{m}) = m/tc^2 + 2/t^2c^4$
  $\Rightarrow$ *correlation time* $t \sim 1/mc^2 = t_C$
- $\langle |\boldsymbol{v}| \rangle|_{t=t_C} = (\langle |\boldsymbol{p}| \rangle / \langle \tilde{m} \rangle)|_{t=t_C} = c$.
  On distances $L \gg \lambda_C$ the average velocity is $< c$ (as it should!).
- *Feynman–Hibbs scaling:*

  Define $\Delta x \equiv |\langle \boldsymbol{x}'', t'' | \hat{\boldsymbol{x}}(\tau + \Delta t) - \hat{\boldsymbol{x}}(\tau) | \boldsymbol{x}', t' \rangle|$
  with $t' \leq \tau \leq t''$ then for

  $\Delta x \ll \lambda_C$ we have $\Delta x \propto \Delta t \quad \Rightarrow \quad d_H = 1$ while for
  $\Delta x \gg \lambda_C$ we have $\Delta x \propto \sqrt{\Delta t} \quad \Rightarrow \quad d_H = 2$.

Important scales:

Some simple consequences:

- $\langle \tilde{m} \rangle = m + 1/tc^2$ and $\mathrm{var}(\tilde{m}) = m/tc^2 + 2/t^2c^4$
  $\Rightarrow$ *correlation time* $t \sim 1/mc^2 = t_C$
- $\langle |\mathbf{v}| \rangle|_{t=t_C} = (\langle |\mathbf{p}| \rangle / \langle \tilde{m} \rangle)|_{t=t_C} = c.$
  On distances $L \gg \lambda_C$ the average velocity is $< c$ (as it should!).
- *Feynman–Hibbs scaling:*

  Define $\Delta x \equiv |\langle \mathbf{x}'', t'' | \hat{\mathbf{x}}(\tau + \Delta t) - \hat{\mathbf{x}}(\tau) | \mathbf{x}', t' \rangle|$
  with $t' \leq \tau \leq t''$ then for

  $\Delta x \ll \lambda_C$ we have $\Delta x \propto \Delta t \quad \Rightarrow \quad d_H = 1$ while for
  $\Delta x \gg \lambda_C$ we have $\Delta x \propto \sqrt{\Delta t} \quad \Rightarrow \quad d_H = 2.$

Important scales:

Some simple consequences:

- $\langle \tilde{m} \rangle = m + 1/tc^2$ and $\text{var}(\tilde{m}) = m/tc^2 + 2/t^2c^4$
  $\Rightarrow$ *correlation time* $t \sim 1/mc^2 = t_C$
- $\langle |\boldsymbol{v}| \rangle|_{t=t_C} = (\langle |\boldsymbol{p}| \rangle / \langle \tilde{m} \rangle)|_{t=t_C} = c$.
  On distances $L \gg \lambda_C$ the average velocity is $< c$ (as it should!).
- *Feynman–Hibbs scaling:*

  Define $\Delta x \equiv |\langle \boldsymbol{x}'', t''|\hat{\boldsymbol{x}}(\tau + \Delta t) - \hat{\boldsymbol{x}}(\tau)|\boldsymbol{x}', t'\rangle|$
  with $t' \leq \tau \leq t''$ then for

  $\Delta x \ll \lambda_C$ we have $\Delta x \propto \Delta t \quad \Rightarrow \quad d_H = 1$ while for
  $\Delta x \gg \lambda_C$ we have $\Delta x \propto \sqrt{\Delta t} \quad \Rightarrow \quad d_H = 2$.

Important scales:

Some simple consequences:

- $\langle \tilde{m} \rangle = m + 1/tc^2$ and $\mathrm{var}(\tilde{m}) = m/tc^2 + 2/t^2c^4$
  $\Rightarrow$ *correlation time* $t \sim 1/mc^2 = t_C$
- $\langle |\boldsymbol{v}| \rangle|_{t=t_C} = (\langle |\boldsymbol{p}| \rangle / \langle \tilde{m} \rangle)|_{t=t_C} = c$.
  On distances $L \gg \lambda_C$ the average velocity is $< c$ (as it should!).
- *Feynman–Hibbs scaling:*

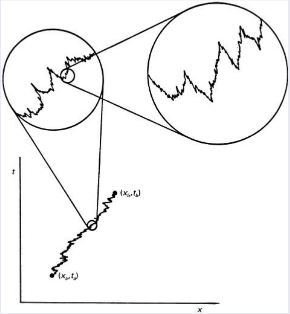
  Define $\Delta x \equiv |\langle \boldsymbol{x}'', t'' | \hat{\boldsymbol{x}}(\tau + \Delta t) - \hat{\boldsymbol{x}}(\tau) | \boldsymbol{x}', t' \rangle|$
  with $t' \leq \tau \leq t''$ then for

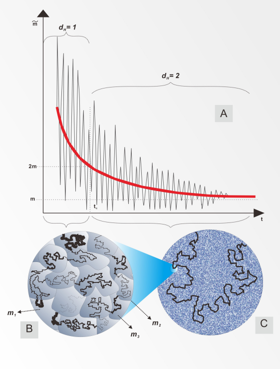
  $\Delta x \ll \lambda_C$ we have $\Delta x \propto \Delta t \;\;\Rightarrow\;\; d_H = 1$ while for
  $\Delta x \gg \lambda_C$ we have $\Delta x \propto \sqrt{\Delta t} \;\;\Rightarrow\;\; d_H = 2$.

Path roughness

*Roughness of representative trajectories in superstat. PI**

Representative trajectory in the usual PI





* P.J. and F. Scardigli, Int. J. Mod. Phys. B **26** (2012) 1241003; PRD **86** (2012) 025029

Outline

1. *From QM to stochastic processes and back*
   - Path-integral appetizer
2. *Superstatistics paradigm*
   - Old wine in new bottles
   - Chapman–Kolmogorov eq. for Markov process
3. *Applications to relativistic QM*
   - Klein–Gordon particle
   - Emergent relativity
   - Gravity

Gravity

When the spacetime is curved, a metric tensor enters in both PI's in a different way, yielding different *"counterterms"*.

In Bastianelli–van Nieuwenhuizen's time slicing regularization scheme

$$\frac{\boldsymbol{p}^2}{2\tilde{m}} \;\mapsto\; \frac{g^{ij}p_i p_j}{2\tilde{m}} + \frac{\hbar^2}{8\tilde{m}}\left(R + g^{ij}\Gamma^m_{il}\Gamma^l_{jm}\right),$$

$$\sqrt{\boldsymbol{p}^2 + m^2c^2} \mapsto \sqrt{g^{ij}p_i p_j + \frac{\hbar^2}{4}\left(R + g^{ij}\Gamma^m_{il}\Gamma^l_{jm}\right) + m^2c^2}$$
$$+ \; \hbar^4\Phi(R, \partial R, \partial^2 R,) \; + \; \mathcal{O}(\hbar^6)$$

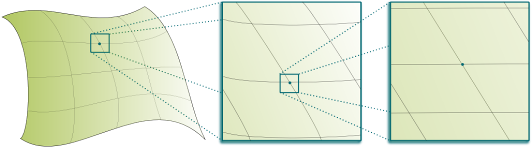
$\Rightarrow$ superstatistics PI identity breaks down **!!**

$\Rightarrow$ respective two cases will lead to *different* physics.

Gravity

Einstein's *equivalence principle* requires that the *local spacetime* structure can be identified with the *Minkowski spacetime* possessing *local Lorentz symmetry*

⇒ one might assume the validity of SPI's at least locally.



The characteristic size of the local inertial frame is of order $1/|K|^{1/4}$ where $K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is the *Kretschmann invariant*.

⇒ breakdown of SPI' happens when $\lambda_C \gtrsim 1/|K|^{1/4}$.*

* P.J. and F. Scardigli, EPJC 73 (2013) 2491

Summary

- Fusion of PI's calculus with superstatistics allows to address
  1 multi-scale stochastic processes
  2 new classes of PI $\Rightarrow$ applications in relativistic QM.

- Wiener process on the fast-time scale is the most plane stochastic process (no memory, directional democracy, etc.)

- Inhomogeneous or polycrystalline vacuum turns the local Galileo symmetry to emergent Lorentz symmetry.

- Non-trivial implications for relativistic QM in curved spaces, e.g., charge-parity violation*.

* P.J. and F. Scardigli, EPJC 73 (2013) 2491

Epilogue

"It would be so nice if something would make sense for a change"

- Alice, *Through the Looking Glass* (Lewis Carroll)

Gravity

Examples:

- *Schwarzschild geometry:* $K = 12\, r_s^2/r^6$

$\Rightarrow$ breakdown should be expected at radial distances $r \lesssim (\lambda_C^2 r_s)^{1/3}$ which are deeply buried below the Schwarzschild event horizon.

- *Robertson–Walker geometry:* $K = 12\,(\dot{a}^4 + a^2\ddot{a}^2)/(ac)^4$

$\Rightarrow$ breakdown should be expected at $(\dot{a}^4 + a^2\ddot{a}^2) \gtrsim (ac/\lambda_C)^4$ ($a(t)$ is the RW scale factor)

In *Vilenkin–Ford model* for inflationary cosmology, $a(t) = A\sqrt{\sinh(Bt)}$ with $B = 2c\sqrt{\Lambda/3}$ (Λ is the cosmological constant), we obtain

$$t \lesssim \frac{1}{B}\,\mathrm{arcsinh}\left[\frac{B\lambda_C}{\left(8c^4 - (B\lambda_C)^4\right)^{1/4}}\right] \equiv \bar{t}$$

Gravity

With $\Lambda \simeq 10^{-52}\text{m}^{-2}$ and τ-lepton Compton's wavelength $\lambda_C^\tau \simeq 6.7 \times 10^{-16}\text{m}$ (yielding the tightest upper bound on t)

$$\Rightarrow \ \bar{t} \simeq 4 \times 10^{-24}\text{s}$$

NOTE: There is no unified theory of particles and antiparticles in the *non-relativistic physics* — formally one has two separate theories.

The ensuing matter-antimatter asymmetry might be relevant in the early Universe, e.g., for *leptogenesis**.

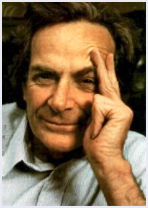
NOTE: $\bar{t}$ is consistent with the *nonthermal leptogenesis* period: 10^{-26}–10^{-12}s after BB.

* P.J. and F. Scardigli, PRD **86** (2012) 025029; arXiv:1301.4091 [hep-th], EPJC (2013)

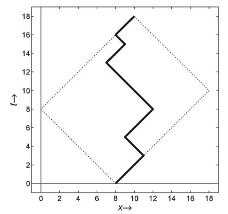
Path integral without action

There is close a connection with *Feynman's checkerboard* PI

Feynman's checkerboard path integral:



R.P. Feynman
(1918-1988)



The dotted rectangle indicates the region of contributing paths

$$G = \sum_{\text{zig-zag paths}} \left(i \frac{t}{N} \frac{mc^2}{\hbar} \right)^R = \sum_R \phi^{(N)}(R) \left(i \frac{t}{N} \frac{mc^2}{\hbar} \right)^R$$

R is number of reversals of zig-zag paths

G is the *propagator* for $1+1$ dimensional Dirac's equation

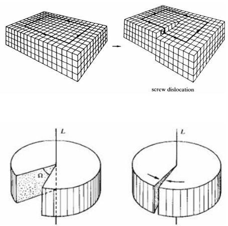
$$i\hbar \frac{\partial \psi}{\partial t} = mc^2 \sigma_x \psi - ic\hbar \sigma_z \frac{\partial \psi}{\partial x}$$

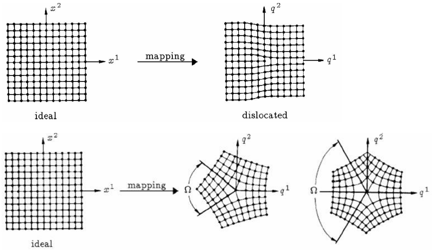
Discrete lattice and generalization to GR

$\mathcal{Q}$: Is there a natural way to extend our picture to GR?

$\mathcal{A}$: Yes, geometry of Einstein and Einstein-Cartan spaces can be considered as being a manifestation of the defect structure of a crystal whose lattice spacing is of the order of ℓ_p — "world crystal" *

- Curvature is due to rotational def., torsion due to translational def.





* H. Kleinert, 2010 textbook, (WS); G. Volovich et all, Ann. Phys. 216 (1992), R. Jackiw et all, Ann. Phys. 308 (2003)

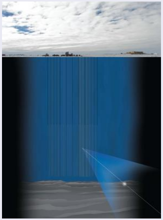
Discrete lattice and GR

Burgers vector (dislocation)	$T \neq 0$	$R = 0$
Frank vector (disclination)	$T = 0$	$R \neq 0$

- At long distances the memory of the crystalline structure is lost.

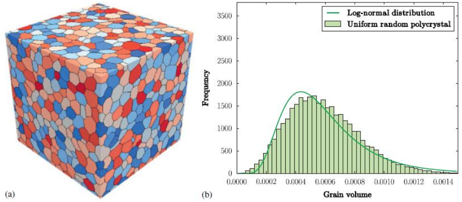
Discrete-spacetime probes:

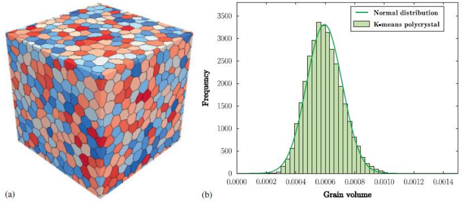
Planck surveyor 14.02.2010



IceCube probe 2010/11

Deformed special relativity — robustness





Different distributions of volumes of Voronoi cells in polycrystalline.

Deformed special relativity — robustness

Understanding the *robustness* of the emergent SR under small variations in ω can guide the study of the relation between SR and deformed SR. To this end we explore connection between $\delta\omega$ and δF, i.e.

$$e^{-t[F(p_v)+\delta F(p_v)]} = \int_0^\infty \mathrm{d}v\, e^{-vp_v}[\omega(v,t)+\delta\omega(v,t)]$$

Requiring that the new smearing PDF is again positively skewed with $v \in \mathbb{R}^+$, and seeking δv in the form

$$\delta v(v,t) = v^\alpha \sum_{n=0}^\infty \epsilon_n(t) v^n, \qquad \alpha \le 1, \quad \epsilon_n(t) \ll 1$$

one arrives at $\delta F(s)$ which admits Laurent expansion in powers of $\sqrt{s}$. If we truncate the expansion after ϵ_1

$$\bar{H} = \epsilon_1/4 + (1+\epsilon_0/2)\sqrt{\boldsymbol{p}^2 c^2 + m^2 c^4 + \epsilon_2/4}$$

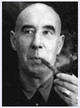
with $\epsilon_1 = -2(1+\epsilon_0/2)\sqrt{\epsilon_2}$. This is Magueijo–Smolin's DSR.

Troubles with Newton–Wigner PI

NOTE: "$\sqrt{}$" PI representation is known as the *Newton-Wigner propagator*.

Problem:

True relativistic propagator must include also *negative* energy spectrum, reflecting the existence of charge-conjugated solutions — *antiparticles*. (Stückelberg, 1935)



Solution: Write equation for Klein-Gordon particle in Schrödinger-like form — *Feshbach-Villars representation*

$$i\partial_t\Psi \;=\; H_{\rm FV}(\boldsymbol{p})\Psi \quad \text{and} \quad H_{\rm FV}(\boldsymbol{p}) \;=\; (\sigma_3 + i\sigma_2)\frac{\boldsymbol{p}^2}{2m} \;+\; \sigma_3 mc^2$$

where Ψ is a two component object

Feshbach–Villars representation

NOTE: The doubling of the fields implies the simultaneous description of particles and antiparticles.

Hamiltonian $H_{\mathrm{FV}}(\boldsymbol{p})$ can be diagonalized as

$$H_{\mathrm{FV}}(\boldsymbol{p}) = U(\boldsymbol{p}) \begin{pmatrix} c\sqrt{\boldsymbol{p}^2 + m^2c^2} & 0 \\ 0 & -c\sqrt{\boldsymbol{p}^2 + m^2c^2} \end{pmatrix} U(\boldsymbol{p})^{-1}$$

$$\equiv U(\boldsymbol{p})\sigma_3 U(\boldsymbol{p})^{-1} H(\boldsymbol{p})$$

U is non-unitary hermitian matrix

$$U(\boldsymbol{p}) = \frac{(1+\gamma_v) + (1-\gamma_v)\sigma_1}{2\sqrt{\gamma_v}}$$

Green's function $\mathcal{G}(x, y)$ associated with the F-V Schrödinger equation is

$$(i\partial_t - H_{\mathrm{FV}})\mathcal{G}(\boldsymbol{x}, t; \boldsymbol{x}', t') = i\delta^{(D)}(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')$$

Feshbach–Villars representation

The solution can be written as

$$\mathcal{G}(x;y) \;=\; \frac{i}{c^2}\int_{\mathbb{R}^4}\frac{d^{D+1}p}{(2\pi)^{D+1}}\,\frac{e^{-ip(x-y)}}{p^2-m^2c^2+i\epsilon}\left[p_0\,c+(\sigma_3+i\sigma_2)\frac{\boldsymbol{p}^2}{2m}+\sigma_3 mc^2\right]$$

$i\epsilon$ prescription $\Rightarrow$ *Feynman boundary condition*

NOTE: Imaginary-time Green func. $\mathcal{G}(\boldsymbol{x},-it;\boldsymbol{x}',-it') \equiv P(\boldsymbol{x},t|\boldsymbol{x}',t')$ is a solution of the *Fokker–Planck like* equation

$$(\partial_t + H_{\mathrm{FV}})P(\boldsymbol{x},t|\boldsymbol{x},t') \;=\; \delta(t-t')\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$$

$$\text{where}\quad P(\boldsymbol{x},t|\boldsymbol{x}',t') = \langle\boldsymbol{x}|e^{-(t-t')H_{\mathrm{FV}}}|\boldsymbol{x}'\rangle$$

$$\text{or}\quad P(\boldsymbol{x},t|\boldsymbol{x}',t') = \int_{\mathbb{R}^D}\frac{d\boldsymbol{p}}{(2\pi)^D}\,e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}'')}\,U(\boldsymbol{p})\langle\boldsymbol{x}''|e^{-(t-t')\sigma_3 H}|\boldsymbol{x}'\rangle U(\boldsymbol{p})^{-1}$$

Feshbach–Villars representation

Warning:

We *cannot* write naively

$$\langle \boldsymbol{x}| e^{-(t-t')\sigma_3 H_{\boldsymbol{p}}} |\boldsymbol{x}'\rangle \quad = \quad \int_{\boldsymbol{x}(t')=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}} \mathcal{D}\boldsymbol{x}\, \frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D}\, e^{\int_{t'}^{t} d\tau \left[i\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - c\sigma_3\sqrt{\boldsymbol{p}^2+m^2c^2}\right]}$$

it diverges for the lower components of the evolution operator

$$e^{-t\sigma_3 H_{\boldsymbol{p}}} = \begin{pmatrix} e^{-tH} & 0 \\ 0 & e^{tH} \end{pmatrix}$$

This can be circumvented by forming *superpositions* of integrals which differ for upper and lower components of $\exp(-t\sigma_3 H(\boldsymbol{p}))$

Feshbach–Villars representation

In particular

$$\langle \boldsymbol{x}|e^{-t\sigma_3 \hat{H}_{\boldsymbol{p}}}|\boldsymbol{x}'\rangle \;=\; \int_0^\infty dv\,\omega(v,t)\int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}} \mathcal{D}\boldsymbol{x}\,\frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D}\,e^{\int_0^t d\tau\left[i\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - v(\boldsymbol{p}^2c^2+m^2c^4)\right]}$$

The weight function is a matrix valued *Weibull distribution*:

$$\omega(v,t) \;=\; \frac{1}{2\sqrt{\pi}\sqrt{v^3/|t|}}\begin{pmatrix} \theta(t)\,e^{-t/4v} & 0 \\ 0 & \theta(-t)\,e^{t/4v}\end{pmatrix}$$

Related notes:

- By going back to real times, we recover Green's function associated with the F-V Schrödinger equation.
- Weibull's PDF brought us automatically into a *Polyakov gauge*.
- Dirac particle in *Foldy–Wouthuysen* rep. can be treated alike.

* P.J. and H. Kleinert, arXiv:1007.3922

Generalizations

More formal statement:

Stochastic process described by the Kramers–Moyal equation with the relativistic Hamiltonian $c\sqrt{\boldsymbol{p}^2 + m^2c^2}$ is equivalent to a doubly stochastic process in which the fast-time dynamics of a free non-relativistic particle (Brownian motion) is coupled with the long-time dynamics describing fluctuations of particle's Newtonian mass.

NOTE: Above conclusions extend also to Dirac's Hamiltonian

$$H_{\rm D}^{A,V} \ = \ c\gamma_0\boldsymbol{\gamma}\cdot(\boldsymbol{p} - e\boldsymbol{A}/c) \ + \ \gamma_0(mc^2 + V) \ + \ eA_0$$

and to the Feshbach–Villars Hamiltonian

$$H_{\rm FV}^{A,V} \ = \ (\sigma_3 + {\rm i}\sigma_2)\frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A}/c)^2 \ + \ \sigma_3(mc^2 + V) + eA_0$$

Generalizations

E.g., when $V = 0$, $A_x = -By$ ($B_z \equiv B$) and $A_y = A_z = 0$ then PI for Dirac's Hamiltonian yields corresponds to the "fast scale Hamiltonian"

$$H_{\rm SP} \ = \ \frac{1}{2\tilde{m}}\left[\left(p_x + \frac{e}{c}By\right)^2 \ + \ p_y^2 \ + \ p_z^2\right] \ - \ \mu_{\rm B} B\sigma_3$$

This is *Schrödinger-Pauli* Hamiltonian with $\mu_{\rm B} = e\hbar/2\tilde{m}c$ representing *Bohr magneton*.

NOTE 1: Smearing distribution ω stays the same as for a free-particle.

NOTE 2: Analogous reasonings can be performed also for charged spin-0 particles, such as, e.g, $\pi^{\pm}$ mesons.

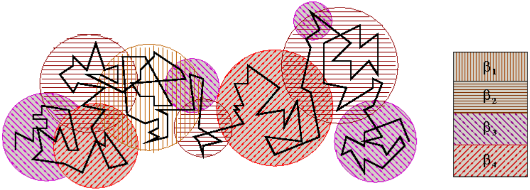
What is so *Super* about Superstatistics?

Superstatistics

C. Beck
(1959 - ∗)

Complex systems often exhibit a dynamics that can be regarded as a superposition of several dynamics on different time scales. ... The two effects produce a superposition of two statistics, or in a short, a *"superstatistics"*.*

* C. Beck et al., Physica **A** 322 (2003); C. Beck, Phys. Rev. Lett. **98** (2007)



Temperature fluctuations in phase-space. Locally is system in equilibrium with temperature β_i.