# Non-commutative probability, conditional expectation values as weak values.

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# Some Surprising Results.

1. von Neumann's 1931 approach is mathematically identical to Moyal's 1949

[von Neumann, Math. Ann. **104** (1931) 570-87]

[Moyal, Proc. Camb. Phil. Soc, **45**, (1949), 99-123]

2. Moyal's conditional expectation values of momentum and energy are intimately related to the energy-momentum tensor $T^{\mu\nu}(\boldsymbol{x},t)$ of standard quantum field theory.

$$\rho(x,t)P_M^j(x,t) = T^{0j}(x,t) \quad \text{and} \quad \rho(x,t)E_M(x,t) = T^{00}(x,t)$$

[Hiley, and Callaghan, arXiv: 1011.4031 and arXiv: 1011.4033.]

3. The Moyal momentum IS the Bohm momentum for the Schrödinger, Pauli and Dirac particles.

[Hiley, and Callaghan, Found. Phys. **42** (2012) 192-208]

4. The Moyal/Bohm approach is about non-commutative probability theory.

[Hiley, arXiv 1211.2098]

5. The Moyal/Bohm momentum is the weak value of the momentum operator,

$$P^{\mu}_{\psi,B}(x,t) = \frac{\langle x|\widehat{P}|\psi(x,t)\rangle}{\langle x|\psi(x,t)\rangle}$$

[Leavens, Found. Phys., **35** (2005) 469-91]
[Wiseman, New J. Phys., **9** (2007) 165-77.]
[Hiley J. Phys.: Conference Series, 361 (2012) 012014.]

6. Bohm momentum, energy, Bohm kinetic energy and hence the quantum potential can be measured using weak values.

[Rob Flack next lecture]

7. Classical physics emerges from a non-commutative statistical (quantum) structure grounded in **process.**

[Hiley, Lecture Notes in Physics, vol. 813, pp. 705-750, Springer (2011).]

# The von Neumann 1931 Algebra.

Start with **translations in *x* and *p*.**

$\widehat{U}(\alpha) = \exp(i\alpha\widehat{P})$ and $\widehat{V}(\beta) = \exp(i\beta\widehat{X})$ $[\widehat{X}, \widehat{P}] = i$

Combine to give $\widehat{S}(\alpha, \beta) = \exp i(\alpha\widehat{P} + \beta\widehat{X})$ Element of Heisenberg group **Radar!**

$\alpha, \beta$ span a symplectic space. Symplectic Clifford algebra E Enveloping algebra of Heisenberg algebra.

[Crumeyrolle, *Orthogonal and Symplectic Clifford Algebras*, 1990]

The symplectic Clifford group is the **metaplectic** group [Guillemin, and Sternberg, Symplectic Techniques in Physics, Cambridge (1984)]

von Neumann shows $\widehat{A} \Leftrightarrow a(\alpha, \beta)$

$$\widehat{A} = \int\int a(\alpha, \beta)\widehat{S}(\alpha, \beta)d\alpha d\beta$$

element of the symplectic space

'symbol'

[von Neumann, Math. Ann. **104** (1931) 570-87]

Formally introduce $\rho_\Psi(x) \to [\,= |\psi\rangle\langle\psi| = \Psi_L(x)\Psi_R(x)]$ and form $F_\Psi(\alpha, \beta) = \mathrm{T}r[\widehat{S}(\alpha, \beta)\rho_\Psi(x)]$

$$\langle\widehat{A}\rangle = \int\int a(\alpha, \beta)F_\Psi(\alpha, \beta)d\alpha d\beta$$

[Moyal, Proc. Camb. Phil. Soc, **45**, (1949), 99-123]

looks like a probability measure?

**Classical expectation value?** Unfortunately $F_\Psi(\alpha, \beta)$ can take negative values. [Bartlett, Math. Proc. Cam. Phil. Soc.,41, (1945) 71-3]. [Groenewold, Physica, XII, (1946) 405-460]

**Don't worry we are in a non-commutative symplectic manifold.** **Non-commutative Probability**

# Non-commutative Phase Space.

**Products of symbols** If $\widehat{C} = \widehat{A}\widehat{B}$ then $C(x,x') = \int A(x,x'')B(x'',x')dx''$

von Neumann shows $c(\alpha,\beta) = a(\alpha,\beta) \star b(\alpha,\beta) = \int\int e^{2i(\gamma\beta-\delta\alpha)}a(\gamma-\alpha,\delta-\beta)b(\alpha,\beta)d\alpha d\beta$

[von Neumann, Math. Ann., **104** (1931) 570-87]

**Moyal product**

Special case: $\alpha\star\beta - \beta\star\alpha = i$

Moyal chose new variables $\alpha \to x \quad \beta \to p$

[Moyal, Proc. Camb. Phil. Soc, **45**, (1949), 99-123]

**Non-commutative Phase Space**

# A Closer look at the Non-commutative Moyal Algebra.

With star product we can form two types of bracket

$$\{a,b\}_{MB} = \frac{a \star b - b \star a}{i\hbar}$$

Moyal bracket

$$\{a,b\}_{BB} = \frac{a \star b + b \star a}{2}$$

Baker bracket

Moyal showed the star product can also be written in the form

$$a(x,p) \star b(x,p) = a(x,p) \exp[i\hbar(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overrightarrow{\partial}_x \overleftarrow{\partial}_p)/2] b(x,p)$$

Easy to show that $x \star p - p \star x = i\hbar$

Then we have

$$\{a,b\}_{MB} = 2a(x,p) \sin[\hbar(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overrightarrow{\partial}_x \overleftarrow{\partial}_p)/2] b(x,p)$$

$$\{a,b\}_{BB} = a(x,p) \cos[\hbar(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overrightarrow{\partial}_x \overleftarrow{\partial}_p)/2] b(x,p)$$

The important property of these brackets is they contain the classical limit.

Moyal bracket becomes the Poisson bracket. $\{a,b\}_{MB} = \{a,b\}_{PB} + O(\hbar^2) = [\partial_x a \partial_p b - \partial_p a \partial_x a]$

Baker bracket becomes a simple product $\{a,b\}_{BB} = ab + O(\hbar^2)$

# The Dynamics.

Because of non-commutativity

$$H(x,p)\star F_\psi(x,p,t) = i(2\pi)^{-1}\int e^{-i\tau p}\psi^*(x-\tau/2)\overrightarrow{\partial}_t\psi(x+\tau/2)d\tau$$

$$F_\psi(x,p,t)\star H(x,p) = -i\int e^{-i\tau p}\psi^*(x-\tau/2)\overleftarrow{\partial}_t\psi(x+\tau/2)$$

Subtracting gives **Moyal bracket** equation

$$\partial_t F_\psi = (H\star F_\psi - F_\psi\star H)/2i = \{H,F_\psi\}_{MB}$$

**Classical Liouville equation to** $O(\hbar^2)$

$\tau\rightarrow\hbar\tau$

Adding gives **Baker bracket** equation $\{H,F_\psi\}_{BB} = (H\star F_\psi + F_\psi\star H)/2$

$$2\{H,F\}_{BB} = i(2\pi)^{-1}\int e^{-i\tau p}[\psi^*(x-\tau/2)\overrightarrow{\partial}_t\psi(x+\tau/2) - \psi^*(x-\tau/2)\overleftarrow{\partial}_t\psi(x+\tau/2)d\tau$$

Writing $\psi = Re^{iS}$ we obtain

$$\frac{\psi^*\overleftrightarrow{\partial}_t\psi}{\psi^*\psi} = \frac{[\psi^*\overrightarrow{\partial}_t\psi - \psi^*\overleftarrow{\partial}_t\psi]}{\psi^*\psi} = \left[\frac{\partial_t R(x+\tau/2)}{R(x+\tau/2)} - \frac{\partial_t R(x-\tau/2)}{R(x-\tau/2)}\right] + i\left[\frac{\partial_t S(x+\tau/2)}{S(x+\tau/2)} + \frac{\partial_t S(x-\tau/2)}{S(x-\tau/2)}\right]$$

Go to the limit $O(\hbar^2)$

$$H\star F_\psi + F_\psi\star H = -2(\partial_t S)F_\psi + O(\hbar^2) \quad\Rightarrow\quad 2(\partial_t S)F_\psi + \{H,F_\psi\}_{BB} = 0$$

$$\frac{\partial S}{\partial t} + H = 0$$

**Classical H-J equation.**

**No need for decoherence to reach the classical level**

# Time Development Equations.

| $X-P$ Phase Space | | |
|---|---|---|
| $\frac{\partial F}{\partial t}+[F,H]_{MB}=0$ | | |
| $2\frac{\partial S}{\partial t}F+[F,H]_{BB}=0$ | | |
| ↑<br>**von Neumann/Moyal algebra** | | |

## Surprise Number 2.

Moyal asked: if $F_\psi(x,p)$ is treated as a probability distribution

What is the conditional expectation value of the momentum?

$$\rho(x)\bar{p} = \int pF_\psi(x,p)dp = \left(\frac{1}{2i}\right)[(\partial_{x_1} - \partial_{x_2})\psi(x_1)\psi(x_2)]_{x_1=x_2=x}$$

**Moyal momentum**

With $\psi = Re^{iS}$, we find $\bar{p}(x) = \frac{1}{2i}[\psi^*\nabla\psi - (\nabla\psi^*)\psi] = \nabla S$

**Moyal momentum = Bohm momentum.**

Moyal's transport of $\bar{p}$

$$\partial_t(\rho\bar{p}_k) + \sum_i \partial_{x_i}(\rho p_k \partial_{x_i} H) + \rho\partial_{x_k} H = 0$$

Again with $\psi = Re^{iS}$, we find

$$\frac{\partial}{\partial x_k}\left[\frac{\partial S}{\partial t} + H - \frac{\nabla^2\rho}{8m\rho}\right] = 0$$

**Quantum potential.**

Or

$$\frac{\partial S}{\partial t} + H - \frac{\nabla^2\rho}{8m\rho} = \frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V - \frac{1}{2m}\frac{\nabla^2 R}{R} = 0$$

**Quantum Hamilton-Jacobi equation.**

For details see appendix of Moyal's paper.

[Moyal, Proc. Camb. Phil. Soc. **45**, (1949), 99-123.]

[Hiley, Proc. Int. Conf. Quantum Theory: Reconsideration of Foundations **2**, (2003) 267-86,]

# Time Development Equations.

| $X-P$ Phase Space | Bohm Phase Space | Configuration Space. |
|---|---|---|
| $\frac{\partial F}{\partial t}+[F,H]_{MB}=0$ | $\frac{\partial P_x}{\partial t}+\nabla_x.\left(P_x\frac{\nabla_x S_x}{m}\right)=0$<br>$p_B=\nabla_x S_x$ | |
| $2\frac{\partial S}{\partial t}F+[F,H]_{BB}=0$ | $\frac{\partial S_x}{\partial t}+\frac{1}{2m}\left(\frac{\partial S_x}{\partial x}\right)^2+Q_x+V=0$ | |
| ↑<br>**von Neumann/Moyal algebra** | ↑<br>**Moyal/Bohm** | |

# Time Development of $\rho_\Psi(x,t)$ in Configuration Space.

For a pure state $\rho_\Psi(x,t)$ is idempotent and of rank one.

In the symplectic non-commutative algebra we can form $\rho_\Psi = \Psi_L\Psi_R$; with $\Psi_L \in \mathcal{I}_L$ and $\Psi_R \in \mathcal{I}_R$

**Elements of a left and right ideal.**

Heuristic argument:-

$$|\psi\rangle\langle\psi| \;\Rightarrow\; \widehat{\rho} = A\rangle\langle B \;\rightarrow A\epsilon B = \Psi_L\Psi_R = \rho_\Psi$$

Standard ket $A\rangle$

Idempotent $\rangle\langle$

[Dirac, Math. Proc.Cam. Phil. Soc., B (1939) 416-418]

[Dirac, Quantum Mechanics 3rd Edition p. 79 (1947)]

[Hiley Lecture Notes in Physics, vol. 813, pp. 705-750, Springer (2011)]

As before we have two time development equations

$$i\Psi_R(\overrightarrow{\partial}_t\Psi_L) = \Psi_R(\overrightarrow{H}\Psi_L) \quad \text{and} \quad -i(\Psi_R\overleftarrow{\partial}_t)\Psi_L = (\Psi_R\overleftarrow{H})\Psi_L$$

which we combine by adding and subtracting to find;-

$$i\left[(\overrightarrow{\partial}_t\Psi_L)\Psi_R + \Psi_L(\Psi_R\overleftarrow{\partial}_t)\right] = \left(\overrightarrow{H}\Psi_L\right)\Psi_R - \Psi_L\left(\Psi_R\overleftarrow{H}\right) \; = [H,\rho]_- \qquad \text{(A)}$$

$$i\left[(\overrightarrow{\partial}_t\Psi_L)\Psi_R - \Psi_L(\Psi_R\overleftarrow{\partial}_t)\right] = \left(\overrightarrow{H}\Psi_L\right)\Psi_R + \Psi_L\left(\Psi_R\overleftarrow{H}\right) \; = [H,\rho]_+ \qquad \text{(B)}$$

From (A) we obtain $i\partial_t\rho = [H,\rho]_-$ **Liouville equation**

**Conservation of Probability**

From (B) we obtain

$$i\Psi_R\overleftrightarrow{\partial}_t\Psi_L = [H,\rho]_+$$

**New equation?**

[Brown, and Hiley, 2000, quant-ph/0005026.]

[Brown, Ph.D. Thesis, University of London, 2004.]

**Conservation of Energy.**

## Time Development Equations.

| $X-P$ Phase Space | Bohm Phase Space | Configuration Space |
|---|---|---|
| $\frac{\partial F}{\partial t}+[F,H]_{MB}=0$ | $\frac{\partial P_x}{\partial t}+\nabla_x.\left(P_x\frac{\nabla_x S_x}{m}\right)=0$ | $i\frac{\partial \rho}{\partial t}+[\rho,H]_{-}=0$ |
|  | $p_B=\nabla_x S_x$ |  |
| $2\frac{\partial S}{\partial t}F+[F,H]_{BB}=0$ | $\frac{\partial S_x}{\partial t}+\frac{1}{2m}\left(\frac{\partial S_x}{\partial x}\right)^2+Q_x+V=0$ | $2\frac{\partial S}{\partial t}\rho+[\rho,H]_{+}=0$ |
| ↑ | ↑ | ↑ |
| **von Neumann/Moyal algebra** | **Bohm model** | **Quantum algebra** |

# Project Quantum Algebraic Equations into a Representation.

Project into representation using $P_a = |a\rangle\langle a|$

$$i\frac{\partial P(a)}{\partial t} + \langle[\rho, H]_-\rangle_a = 0$$

$$2P(a)\frac{\partial S}{\partial t} + \langle[\rho, H]_+\rangle_a = 0$$

Choose $P_x = |x\rangle\langle x|$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{K\hat{x}^2}{2}$$

Harmonic oscillator

$$\frac{\partial P_x}{\partial t} + \nabla_x\cdot\left(P_x\frac{\nabla_x S_x}{m}\right) = 0$$

$$\frac{\partial S_x}{\partial t} + \frac{1}{2m}\left(\frac{\partial S_x}{\partial x}\right)^2 + \frac{Kx^2}{2} - \frac{1}{2mR_x}\left(\frac{\partial^2 R_x}{\partial x^2}\right) = 0$$

Conservation of probability

Quantum Hamilton-Jacobi equation.

[M. R. Brown & B. J. Hiley, quant-ph/0005026]

**But there is more!**

Choose $P_p = |p\rangle\langle p|$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{K\hat{x}^2}{2}$$

**Quantum potential**

$$\frac{\partial P_p}{\partial t} + \nabla_p\cdot\left(P_p\frac{\nabla_p S_p}{m}\right) = 0$$

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + \frac{K}{2}\left(\frac{\partial S_p}{\partial p}\right)^2 - \frac{K}{2R_p}\left(\frac{\partial^2 R_p}{\partial p^2}\right) = 0$$

**Possibility of Bohm model in momentum space.**

But now $x = -\left(\frac{\partial S_p}{\partial p}\right)$

Trajectories from the streamlines of probability current. $j_p = -\langle p|\frac{\partial(\hat{\rho}V(\hat{x}))}{\partial x}|p\rangle$

**What is called "Bohmian Mechanics" is but a fragment of the deeper non-commutative geometry.**

# Surprise Number 3:- Energy-Momentum Tensor.

$$T^{\mu\nu} = -\left\{\frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)}\partial^\nu\psi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi^*)}\partial^\nu\psi^*\right\}$$

Take the Schrödinger Lagrangian: $\mathcal{L} = -\frac{1}{2m}\nabla\psi^*\cdot\nabla\psi + \frac{i}{2}[\psi^*(\partial_t\psi) - (\partial_t\psi^*)\psi] - V\psi^*\psi.$

and find

$$T^{0\mu} = -\frac{i}{2}[(\partial^\mu\psi^*)\psi - \psi^*(\partial^\mu\psi)] = \frac{i}{2}[\psi^*\overleftrightarrow{\partial}^\mu\psi] = -\rho\partial^\mu S$$

Recalling that $P_M(x,t) = P_B(x,t) = \nabla S(x,t)$ and $E_M(x,t) = E_B(x,t) = -\partial_t S(x,t)$

Then explicitly:

$\rho(x,t)P_M^j(x,t) = T^{0j}(x,t)$ and $\rho(x,t)E_M(x,t) = T^{00}(x,t)$

These are the LOCAL expressions for the energy-momentum of the particle.

Conservation of energy is maintained through the quantum Hamilton-Jacobi equation.

Similar relations hold for the Pauli and Dirac particles.

**Use orthogonal Clifford algebra.**

[Hiley, and Callaghan, arXiv: 1011.4031 and arXiv: 1011.4033.]

Standard QFT deals with the GLOBAL expression of energy-momentum

$$P^j = \int T^{0j}(x,t)d^3x \qquad E = \int T^{00}(x,t)d^3x$$

# Surprise Number 4:-Weak Values.

We will show that these quantities are related to weak values through:

$$\rho P_{jB} = \rho\partial_j S = -T^{0j} = \Re[i\rho\langle P_j\rangle_W]$$ **Moyal/Bohm momentum.**

$$\rho E_B = -\rho\partial_t S = -T^{00} = \Re[i\rho\langle P_t\rangle_W]$$ **Moyal/Bohm energy.**

What is a weak value? $$A_W = \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}$$ N.B. $A_W \in \mathbb{C}$

[Aharonov and Vaidman, Phys. Rev. **41**, (1990) 11-19]

How do they appear in the formalism?

$$\langle\psi|A|\psi\rangle = \sum\langle\psi|\phi_j\rangle\langle\phi_j|A|\psi\rangle$$ where $|\phi_j\rangle$ form a complete orthonormal set.

Then

**Post select**

$$\langle\psi|A|\psi\rangle = \sum\langle\psi|\phi_j\rangle\left(\frac{\langle\phi_j|\psi\rangle}{\langle\phi_j|\psi\rangle}\right)\langle\phi_j|A|\psi\rangle = \sum\rho_j\frac{\langle\phi_j|A|\psi\rangle}{\langle\phi_j|\psi\rangle}$$

Weak value.

Remember $\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}$ is a complex number. It is clearly a transition probability amplitude.

But how is this related to the energy-momentum component $T^{0\mu}(x,t)$?

# Weak values when $\hat{P}$ is involved.

Weak value $$\langle \boldsymbol{P}\rangle_W = \frac{\langle \boldsymbol{x}|\boldsymbol{P}|\psi(t)\rangle}{\langle \boldsymbol{x}|\psi(t)\rangle}$$

Form: $$\langle \boldsymbol{x}|\hat{\boldsymbol{P}}|\psi(t)\rangle = \int \langle \boldsymbol{x}|\hat{\boldsymbol{P}}|\boldsymbol{x}'\rangle\langle \boldsymbol{x}'|\psi(t)\rangle d\boldsymbol{x}' = -i\boldsymbol{\nabla}\psi(\boldsymbol{x},t)$$

Write $\psi(\boldsymbol{x},t) = R(\boldsymbol{x},t)e^{iS(\boldsymbol{x},t)}$ then

$$\langle \boldsymbol{P}\rangle_W = \boldsymbol{\nabla}S(\boldsymbol{x},t) - i\boldsymbol{\nabla}\rho(\boldsymbol{x},t)/2\rho(\boldsymbol{x},t)$$ with $\rho(\boldsymbol{x},t) = |\psi(\boldsymbol{x},t)|^2$

Moyal/Bohm momentum. osmotic momentum.

Real part of weak value:

$$\Re[i\rho\langle \boldsymbol{P}\rangle_W] = [\boldsymbol{\nabla}\psi^*(\boldsymbol{x})]\psi(\boldsymbol{x}) - \psi^*(\boldsymbol{x})[\boldsymbol{\nabla}\psi(\boldsymbol{x})] = \psi^*(\boldsymbol{x})\overleftrightarrow{\boldsymbol{\nabla}}\psi(\boldsymbol{x}) = \rho\boldsymbol{P}_B$$

**Moyal/Bohm momentum**
$\boldsymbol{T^{0j}(x,t)}$

Imaginary part of weak value: $$\Im[-i\rho\langle \boldsymbol{P}\rangle_W] = [\boldsymbol{\nabla}\psi^*(\boldsymbol{x})]\psi(\boldsymbol{x}) + \psi^*(\boldsymbol{x})[\boldsymbol{\nabla}\psi(\boldsymbol{x})] = \boldsymbol{\nabla}[\rho(\boldsymbol{x})].$$

[Bohm and Hiley, Phys. Reps, **172**, (1989) 92-122.]

## The Bohm kinetic energy.

$$\Re[\langle \boldsymbol{P}^2\rangle_W] = (\boldsymbol{\nabla}S(\boldsymbol{x}))^2 - \frac{\boldsymbol{\nabla}^2 R(\boldsymbol{x})}{R(\boldsymbol{x})} = P_B^2 + Q.$$

$$\Im[\langle \boldsymbol{P}^2\rangle_W] = \boldsymbol{\nabla}^2 S(\boldsymbol{x}) + \left(\frac{\boldsymbol{\nabla}\rho(\boldsymbol{x})}{\rho(\boldsymbol{x})}\right)\boldsymbol{\nabla}S(\boldsymbol{x}).$$

[Leavens, Found. Phys., 35 (2005) 469-91]

[Wiseman, New J. Phys., 9 (2007) 165-77.]

[Hiley, J. Phys Conf. Series, **361**, (2012), 012014]

# Bohm Approach and Pauli spin.

Density element $\rho(x) = \phi_L(x)\phi_R(x) \in$ orthogonal Clifford algebra With $\Psi = \begin{pmatrix} R_1 e^{iS_1} \\ R_2 e^{iS_2} \end{pmatrix}$

The Bohm momentum and energy

$$\rho P_B(x) = \rho_1(x)\nabla_x S_1(x) + \rho_2(x)\nabla_x S_2(x) = \Re[i\rho\langle P_j\rangle_W]$$ **Bohm Momentum**

$$\rho E_B(x) = \rho_1(x)\partial_t S_1(x) + \rho_2(x)\partial_t S_2(x) = \Re[i\rho\langle P_t\rangle_W]$$ **Bohm Energy**

The Bohm kinetic energy is

$$\Re[\langle P^2\rangle_W] = P_B^2(x) + [2(\nabla_x W(x)\cdot S(x)) + W^2(x)] = P_B^2 + Q.$$

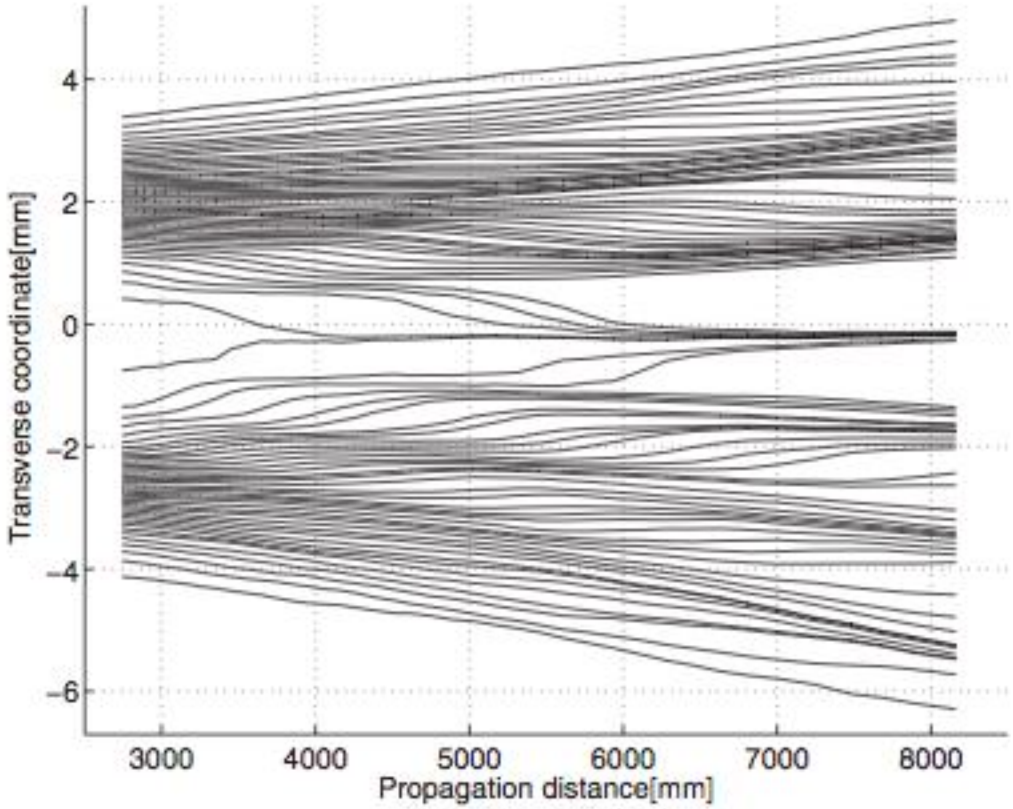
Spin of particle $S = i(\phi_L e_3 \widetilde{\phi_L})$ and $\rho W = \nabla_x(\rho S)$ with $\phi_R(x) = \tilde{\phi}_L$

[Hiley, and Callaghan, Found. Phys. **42**, (2012) 192 and Maths-ph: 1011.4031]

This generalises to the Dirac particle

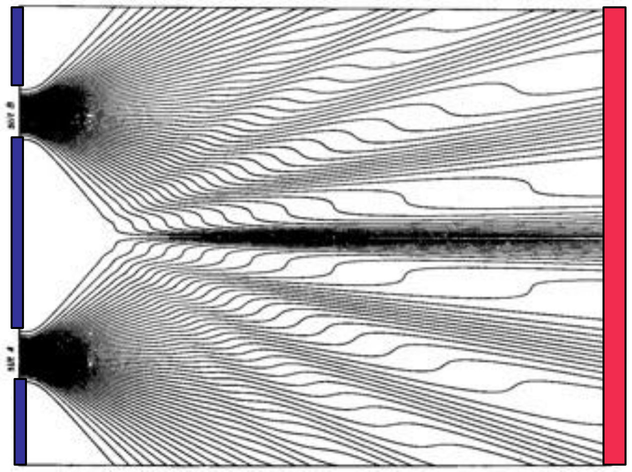
[Hiley and Callaghan, Fond. Phys 42 (2012) 192-208,
math-ph:1011.4031 and 1011.4033]

# Photon 'trajectories'.



Experimental--Photons.

[Kocsis, Braverman,Ravets,Stevens,Mirin, Shalm, Steinberg,
*Science* **332**, 1170 (2011)]



[Philippidis, Dewdney, and Hiley, Nuovo Cimento **52B** (1979) 15-28.]

[Prosser, IJTP , 15, (1976) 169]

Problems with concept of a photon trajectory

We measure $T^{0j}(x,t)$, Poynting's vector.

What is the meaning of the Poynting vector for a single photon?

What is the meaning of a photon at a point?

Must go to field theory

[Bohm, Hiley, and Kaloyerou, Phys. Reports, **144**, (1987) 349-375.]

These criticisms do not apply to non-relativistic particles with finite rest mass (Schrödinger particle)

Need new experiments using atoms.