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Hamiltonian flows, short-time propagators and the quantum Zeno effect

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Abstract. In a recent paper we have examined the short-time propagator for the Schrödinger equation of a point source. An accurate expression modulo Δt^2 for the propagator showed that it was independent of the quantum potential implying that the quantum motion is classical for very short times. In this paper we apply these results to the experiment of Itano, Heinzen, Bollinger and Wineland which demonstrates the quantum Zeno effect in beryllium. We show that the transition is inhibited because the applied continuous wave radiation suppresses the quantum potential necessary for the transition to occur. This shows there is no need to appeal to wave function collapse.

1. Introduction

In a recent paper we have shown that there is a one-to-one correspondence between Hamiltonian flows and the quantum evolution group which only becomes apparent when one uses some of the less well known properties of symplectic geometry [15]. Indeed we have shown that the Schrödinger equation can be *mathematically* derived in a *rigorous* manner without recourse to any physical ‘quantum mechanical’ hypotheses. This result exploits the two-fold covering group of the symplectic group, where the global properties, normally handled by phase changes, can be discussed within the dynamics itself.

When viewed in this way, Bohmian dynamics emerges as a feature of the Hamiltonian flow, where what is normally described by the wave function ψ is now discussed in terms of a Hamiltonian, $H^\psi = H + Q^\psi$. Here H is the usual Hamiltonian operator and Q^ψ is given by

$$Q^\psi = -\frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2}$$

This naturally gives rise to the quantum Hamilton-Jacobi equation¹

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + Q^\psi(x) + V(x) = 0 \quad (1.1)$$

¹ We will only consider a single particle (one dimensional) formalism.



In the standard approach to quantum mechanics, S is interpreted as a phase and $\rho(x)$ is the probability of the particle *being* at the position coordinate x . In our approach the phase is treated as a dynamical variable. The fact that this approach can describe the behaviour of atoms and molecules assumed to be Schrödinger particles is by now well established [5, 22, 13, 10].

In this paper we want to examine the quantum Zeno effect from the perspective of Hamiltonian flows. The reason for considering this effect emerged from a recent paper that was concerned with short-time quantum propagators when analysed in terms of Bohmian dynamics [16]. By studying the short time propagator, we find that any classical potential kicks in immediately i.e. to $O((\Delta t)^2)$. On the other hand the quantum potential does not contribute to the equation of motion to this order. In other words the short time propagator is totally classical in *all* quantum processes. This has a direct consequence for the Zeno effect. In this paper we explain how the Bohm approach explains this phenomenon without appealing to any wave function collapse.

2. The quantum Zeno effect

2.1. The experimental appearance of the Zeno effect

It has been claimed by Misra and Sudarshan [32, 8] that the quantum Zeno effect leads to the conclusion that, and we quote, “an unstable particle observed continuously whether it has decayed or not will *never* be found to decay”. These authors confine their discussion specifically to the case of α -decay in a cloud or bubble chamber. Peres [34] has considered a more general question, namely, “If an unstable quantum system is kept under continuous observation, will it decay?”

Rather than considering α -decay, let us consider the specific example that forms the basis of an actual experiment [25]. This experiment involves the suppression of an atomic transition in beryllium when it is subjected to a series of rf pulses. The key question here is whether these pulses can be regarded as ‘measurements that collapse the wave function’. The answer is in dispute. Itano *et al* [25], Cook [9] and Porrati and Putterman [35] all claim that the pulses do, in fact, induce a ‘collapse’ of the wave function. On the other hand Peres [34] and Block and Berman [3] argue that it is the dynamics of the atom plus the field system alone that suppresses the transition. This is the main question we will address in this paper.

The Bohm approach, as discussed in Bohm and Hiley [5], is based on the assumption that the world has a continuous existence with each particle having simultaneously a definite position and a definite momentum. This momentum is not an eigenvalue of the momentum operator for the state under consideration, but the real part of the weak value of the momentum operator at a point in space. This was shown in detail by Leavens [29] and in Hiley [20]. Furthermore these values, and this is a very significant point, can now be determined by experiment as shown in Flack and Hiley [11].

How then do we explain the suppression of a transition in this approach? Bohm and Hiley [5] offered an explanation but that simply relied on a re-interpretation of the standard quantum formalism. In this paper we use a different approach based on the dynamics of quantum Hamiltonian flows, the details of which have already been discussed in Gosson and Hiley [15].

The basic idea depends upon a general mathematical result reviewed by Guillemin and Sternberg [17]. What they show is that the Schrödinger equation appears naturally as a one-parameter sub-group in the metaplectic group, this group being the double cover of the symplectic group. This argument depends solely on the mathematical structure of the symplectic group and its two-fold cover. This fact has wider implications since it implies that both classical and quantum mechanics depend on the same mathematical object, namely the Hamiltonian flow, viewed as an abstract group. A detailed mathematical proof of this structure has been presented in Gosson and Hiley [15].

When we make the group act on the points of phase space, via the symplectic representation,

we obtain Hamiltonian mechanics, but if it acts on functions, via the metaplectic representation, we obtain quantum mechanics. In both cases we have an associated theory of motion: in the symplectic representation, the motion is governed by Hamilton's equations. In the metaplectic representation the motion is governed by Bohm's equations. Not only does this structure contain the quantum Hamilton-Jacobi equation (1.1), but it also contains the so-called 'guidance condition'

$$p = \nabla S. \quad (2.1)$$

These equations now emerge from the equation of motion governing the time evolution. In adopting the method of Hamiltonian flows, we have, of course, assumed that every particle actually exists at a point in space-time and, furthermore, is assumed to have simultaneously a well defined momentum.

Before jumping to the conclusion that this approach must necessarily be incorrect because it seems to violate the uncertainty principle, we stress once again that the momentum beable is not an eigenvalue of the momentum operator. Not only is it the real part of the weak value of the momentum operator but this value is also related to the $(0, k)^{th}$ component of the energy-momentum tensor, $T^{\mu\nu}(x, t)$ of the quantum field [21] as was first shown by Takabayasi [36] and re-derived and extended to apply also to the Pauli and Dirac particles by Hiley and Callaghan [21]. Using the Hamiltonian flow approach we find this momentum emerges from the dynamics itself.

It should be emphasised that in the method of Hamiltonian flows, we do not adopt the standard view that the wave function is the only way to specify the state of the quantum system. The wave function merely encodes the information about the dynamics of the system. Hence the appearance of the term, Q^ψ , in the quantum Hamilton-Jacobi equation (1.1) is of key significance here since it contains information about the overall experimental environment. Thus although we have a localised object, its behaviour depends on the overall experimental conditions, a point that was strongly emphasised by Bohr [7].

The positive feature of the method of Hamiltonian flows is that, unlike the standard approach, it has no measurement problem. The arguments for this position follow closely the extensive discussions given in Bohm and Hiley [4]. There it is pointed out that there is no need for a collapse of the wave function, the world unfolds in its environment objectively. Thus there is no need for the observer to play any active role in "collapsing the wave function". Of course, if the observer wants to know what has actually happened, he/she must set up a piece of apparatus to display some macroscopic, irreversible effect to reveal which process out of the ensemble of possible processes has actually taken place. This involves making a von Neumann measurement, which merely signifies a change in the wave function, but the world itself evolves regardless of whether we 'look' at the results or not.

In summary then, the main purpose of this paper is to explain the experimentally observed fact [25] that, when we introduce an external process to continuously 'observe' an unstable state, its decay is inhibited.

A preliminary answer to this question has already been sketched out in Bohm and Hiley [6]. Here we use a rigorous mathematical result to confirm that original conjecture. It is our hope that this presentation will help clarify why we feel the method of Hamiltonian flows will provide a clear understanding of this phenomena. Our approach provides support for the conclusion that it is the dynamics of the atom plus the field system that is responsible for the suppression in the quantum transitions thus supporting the conclusions of Peres [34] and of Block and Berman [3].

2.2. The standard approach

Let us first recall the standard argument that shows why a constantly 'observed' or 'watched' transition will not occur. Consider an excited state of some α -active nuclei $|\psi_u(t)\rangle$ and let H

be the Hamilton governing the transition to one of a set of possible decay states $|\phi_{d_i}(t)\rangle$. Then after time t , the state of the nuclei will be given by

$$|\psi(t)\rangle = e^{-iHt}|\psi_u(0)\rangle.$$

The probability that the state remains un-decayed after a time t will be

$$P_u(t) = |\langle\psi_u(t)|e^{-iHt}|\psi_u(0)\rangle|^2 = 1 - \sigma^2 t^2 + O(t^4).$$

Here $\sigma^2 = \langle H^2 \rangle - \langle H \rangle^2$. Thus for a short time interval, $\Delta t = t/n$, the probability that the state is still excited is simply

$$P_u(t/n) \approx 1 - \sigma^2(t/n)^2 + O((t/n)^4).$$

Suppose now we repeat the observation n times so that we have

$$P_u(t) = [P_u(t/n)]^n = [1 - (\sigma t/n)^2]^n.$$

Let us now consider the limit as $n \rightarrow \infty$ by writing

$$\ln P_u(t) = n \ln [1 - (\sigma t/n)^2] = n [-(\sigma t/n)^2 + O(n^{-4})].$$

Then by taking the limit $n \rightarrow \infty$, we see that the state does not decay since $P_u(t) \rightarrow 1$. This result is well known [1], and is sometimes called the ‘watched pot’ or even the ‘watchdog’ effect [27].

Central to this argument is the collapse of the wave function, but as we have already pointed out, there is no collapse in the traditional Bohm approach outlined in [5]. How then is the above result explained in this approach? A preliminary answer has already been presented in Bohm and Hiley [5] where this effect was discussed in the context of the Auger effect. In this paper we will show how these ideas can be used on the actual experimental example first proposed by Cook [9] and later carried out by Itano *et al* [25].

The experiment of Itano *et al* uses three particular atomic states of a beryllium ion placed in a magnetic field. To make our discussion as simple as possible, we will assume level one, $|1\rangle$ is a ground state, level two, $|2\rangle$ is a metastable state with a life time of order 1 sec. The third level, $|3\rangle$ when excited, rapidly relaxes ($\sim 10^{-8}$ sec) only to the ground state emitting a photon in the UV range.

To discuss the principles involved, let us first consider a single Be ion with an electron in $|1\rangle$. The ion is subjected to a optical UV (313nm) source, polarised perpendicular to the applied magnetic field. This causes the electron to jump into level $|3\rangle$. It then quickly relaxes to the ground state emitting a photon, which is then detected. As long as the optical UV source is switched on and the rf source switched off, a continuous burst of optical photons is seen.

Suppose now we switch off the optical source and, instead, switch on a pulse of the rf source lasting time T . This will drive a transition between $|1\rangle$ and $|2\rangle$. If we again start with the atom in state $|1\rangle$, we can choose T so that the probability of the electron being in $|2\rangle$ is $P_2(T) = 1$. If Ω is the Rabi frequency of the transition in the applied magnetic field, the time T will be given by $T = \pi/\Omega$. When the electron ‘jumps’ to the metastable state $|2\rangle$ and the optical pulse is on, no optical photons will be emitted from the atom.

Suppose now with the rf pulse switched on, we allow n short bursts of the optical source to be applied in the time T . The duration of the pulse is chosen to be much shorter than the time between pulses. With equally spaced pulses the time at which the r -th pulse starts is $\tau = rT/n$. Here $r = 1, 2, \dots, n$. What we require to find is the probability $P_2(T)$ of finding level $|2\rangle$ occupied after n such pulses. Using the rotating wave approximation, Itano *et al* [25] find

$$P_2(T) = [1 - \cos^n(\pi/n)]/2$$

Thus $P_2(T) \rightarrow 0$ as $n \rightarrow \infty$. In other words increasing the number of optical pulses stops the electron making a transition into the state $|2\rangle$. Alternatively if the experiment is started with the electron in the state $|2\rangle$, the rf-pulses will prevent the electron relaxing into state $|1\rangle$. In the first case we will detect the emission of a burst of optical photons, while in the second case, no optical photons will be detected. This will allow us to infer in which state the electron has been trapped. Itano *et al* claim that the observation of photons constitutes a “measurement” while the failure to detect any photons constitutes a “null measurement”.

It is clear that we have an observation because we can tell in which state the electron is, but does detecting/not detecting the optical photons actively ‘collapse’ the wave function? This is where a difference of opinion arises. Cook [9], Itano *et al* [25] and Porrati and Putterman [35] all claim that a ‘collapse’ of the wave function has taken place. On the other hand, Peres [34] and Block and Berman [3] claim it is determined solely by the dynamics of the atom plus the field system.

Thus we are faced with the vexed question concerning the exact meaning of the word ‘measurement’ in the context of quantum mechanics. In classical physics, there is an unfolding process which can be observed continuously without disturbing it, or if the means of observation does slightly change the unfolding process, a correction can be made in principle. This is not possible in quantum processes described solely in terms of a wave function. As stressed by Bohr, the word ‘phenomena’ cannot be defined without including “...an account of the whole experimental arrangement” [7] p. 73. In this context, Bohr includes in the experimental conditions a “suitable final amplification device producing, for example, permanent marks on a photographic plate caused by the penetration of electrons into the emulsion”. Without the mark or some other irreversible result, we can only talk about the time development of the wave function.

Since the Schrödinger equation is linear and describes a unitary evolution, the final wave function now includes a linear combination of products of wave functions of the system, $\psi_n(x, t)$ and the apparatus, $\phi_n(y, t)$, viz:

$$\Psi(x, y, t) = \sum c_n(t)\psi_n(x, t)\phi_n(y, t). \quad (2.2)$$

The instrument will be found in one of its eigenstates, say, $\phi_j(y, t)$. Thus we say the wave function has ‘collapsed’. This then defines what Bohr calls “a closed indivisible phenomena”. Whether we can find a dynamical cause for a collapse is, of course, an open question that has received much attention [33] [12] [26], but we will not pursue this discussion in this paper.

When we apply these criteria to the example above, we already see a problem. While the detection of the burst of optical photons provides the necessary irreversible effect, the recording of no photons does not. Nevertheless the linear wave function (2.2) will, in this case, contain the two terms ϕ_{fired} and $\phi_{unfired}$. The collapse argument immediately ‘closes the phenomena’, but without any appeal to irreversibility.

Clearly the recording or otherwise of the optical photons is not where the core of the Zeno effect resides. It is the rapid succession of externally applied continuous wave optical pulses, while the rf-source remains switched on, that traps the electron either in state $|1\rangle$ or $|2\rangle$. But in what sense is this a ‘measurement’? It cannot be argued that the pulses ‘cause’ the wave function to ‘collapse’ into one state or the other, because no non-unitary process is involved. It is for this reason that Block and Berman [3] prefer to talk about the results in terms of the dynamics of the atom plus the field system. In fact they trace the reason for the trapping to the rapid decay of the probability amplitude for the transition $1 \leftrightarrow 2$. This follows from a detailed examination of the full density matrix equations of the motion for the three level system and the two resonant continuous wave fields. Interested readers should consult their paper [3] for the details.

On the other hand, this whole process does constitute a kind of ‘observation’ since it enables one to decide which state the electron is in at any given time. Rather than being a *direct* measurement with wave function collapse, we are essentially making an *inference* about the behaviour of the system. This is much closer to the kind of inference we use in classical physics and, indeed, the kind of inference we make when using the Bohm approach where there is no measurement problem [4]. In this approach one assumes that the system unfolds with the particles following well defined, although unknown, trajectories. Because the trajectories are continuous, it avoids the slippery notion of ‘jumps’ that arise in quantum measurement which this particular example highlights. An extremely enlightening discussion of the measurement problem relevant to our discussion in this paper can be found in Bell [2]. Indeed Bell felt so strongly about the confusion caused by the word ‘measurement’ that he called for a ban on the use of the word when discussing quantum phenomena.

3. The Hamiltonian character of Bohmian trajectories

3.1. The Hamiltonian $H^\psi = H + Q^\psi$

The key feature of the Bohm approach through particle trajectories is to realise the whole process is Hamiltonian. The general case is slightly more subtle and we refer to the papers by Holland [23, 24] for a thorough discussion of the interpretation of Bohmian trajectories from the Hamiltonian point of view. Here we will present a simple, more direct way of demonstrating this fact.

Let us start from the basic dynamical equation of the Bohm approach, namely, the quantum Hamilton-Jacobi equation (1.1). We will now show that this equation leads to Hamilton’s equations of motion provided we introduce the Hamiltonian $H^\Psi = H + Q^\Psi$ so that (1.1) has the form of a classical Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H^\Psi \left(x, \frac{\partial S}{\partial x}, t \right) = 0. \quad (3.1)$$

Differentiating the latter with respect to $p = \partial S / \partial x$ yields, using the chain rule,

$$\frac{\partial^2 S}{\partial x_0 \partial t} + \frac{\partial H^\Psi}{\partial p} \frac{\partial^2 S}{\partial x_0 \partial x} = 0 \quad (3.2)$$

and differentiating the equation (1.1) with respect to time yields

$$\frac{\partial^2 S}{\partial x_0 \partial t} + \frac{\partial^2 S}{\partial x \partial x_0} \dot{x} = 0. \quad (3.3)$$

Subtracting (3.3) from (3.2) we get

$$\frac{\partial^2 S}{\partial x \partial x_0} \left(\frac{\partial H^\Psi}{\partial p} - \dot{x} \right) = 0$$

which produces the first Hamilton equation

$$\dot{x} = \nabla_p H^\Psi(x, p, t).$$

We have assumed that $\partial^2 S / \partial x \partial x_0 \neq 0$ in this derivation. Let us next show that the second Hamilton equation is also satisfied. For this we differentiate the quantum Hamilton–Jacobi equation (1.1) with respect to x , which yields

$$\frac{\partial^2 S}{\partial x \partial t} + \frac{\partial H^\Psi}{\partial x} + \frac{\partial H^\Psi}{\partial p} \frac{\partial^2 S}{\partial x^2} = 0. \quad (3.4)$$

Introducing the canonical momentum $p(t) = \nabla_x S(x(t), x_0; t, t_0)$ and differentiating with respect to t we get

$$\frac{\partial^2 S}{\partial t \partial x} = -\dot{p}(t) - \frac{\partial^2 S}{\partial x^2} \dot{x} \quad (3.5)$$

and hence the equation (3.4) can be rewritten

$$-\dot{p}(t) - \frac{\partial^2 S}{\partial x^2} \dot{x} + \frac{\partial H^\Psi}{\partial x} + \frac{\partial H^\Psi}{\partial p} \frac{\partial^2 S}{\partial x^2} = 0.$$

Taking into account the relation $\dot{x} = \partial H^\Psi / \partial p$ established above we have

$$-\dot{p}(t) - \frac{\partial H^\Psi}{\partial x} = 0$$

which is precisely the second Hamilton equation. This means that the Bohmian trajectory for a particle initially sharply localised at a point x_0 is Hamiltonian, and is governed by the Hamilton equations:

$$\dot{x} = \frac{\partial H^\Psi}{\partial p}(x, p, t) \quad , \quad \dot{p} = -\frac{\partial H^\Psi}{\partial x}(x, p, t). \quad (3.6)$$

In conclusion we have thus shown that:

Bohm's equation of motion (1.1) is equivalent to Hamilton's equations (3.6).

3.2. Short-time propagators

In order to proceed, we need first to study the short-time solutions of the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} (\partial^2 S / \partial x^2) + V(x) \right] \Psi \quad , \quad \Psi(x, t_0) = \delta(x - x_0) \quad (3.7)$$

The solution can be written

$$\Psi(x, t) = \int K(x, x_0; t) \Psi_0(x_0) dx_0 \quad (3.8)$$

where the kernel K is the quantum propagator:

$$K(x, x_0; t) = \langle x | \exp(-i\hat{H}t/\hbar) | x_0 \rangle. \quad (3.9)$$

Schrödinger's equation (3.7) is then equivalent to

$$i\hbar \frac{\partial K}{\partial t} = \hat{H}(x, -i\hbar \partial / \partial x, t) K \quad , \quad K(x, x_0; 0) = \delta(x - x_0) \quad (3.10)$$

where δ is the Dirac distribution. Physically this equation describes an isotropic source of point-like particle emanating from the point x_0 at initial time $t_0 = 0$. We want to find an asymptotic formula for K for short time intervals Δt . Referring to the usual literature, such approximations are given by expressions of the type

$$K(x, x_0; \Delta t) = \left(\frac{1}{2\pi i \hbar} \right)^{1/2} \sqrt{\rho(x, x_0; \Delta t)} \exp \left(\frac{i}{\hbar} S(x, x_0; \Delta t) \right)$$

where $S(x, x_0; \Delta t)$ is the action along the classical trajectory from x_0 to x in time Δt and

$$\rho(x, x_0; \Delta t) = -\frac{\partial^2 S(x, x_0; \Delta t)}{\partial x \partial x_0}$$

is the corresponding Van Vleck density. We will need the precise short-time behaviour of the action, S . In this regard, Makri and Miller [30, 31] have shown that the asymptotic expression for the generating function is given by

$$S(x, x_0; \Delta t) = \frac{m}{2\Delta t}(x - x_0)^2 - \tilde{U}(x, x_0)\Delta t + O(\Delta t^2) \quad (3.11)$$

where $\tilde{U}(x, x_0, 0)$ is the average value of the potential over the straight line joining x_0 at time t_0 to x at time t with constant velocity:

$$\tilde{U}(x, x_0) = \int_0^1 U(\lambda x + (1 - \lambda)x_0, 0)d\lambda. \quad (3.12)$$

Introducing the following notation,

$$\tilde{S}(x, x_0; \Delta t) = m\frac{(x - x_0)^2}{2\Delta t} - \tilde{U}(x, x_0)\Delta t, \quad (3.13)$$

leads us to the Makri and Miller approximation (formula (17c) in [30]) for the short-time propagator:

$$K(x, x_0; \Delta t) = \left(\frac{1}{2\pi i\hbar}\right)^{1/2} \sqrt{\rho(x, x_0; \Delta t)} \exp\left(\frac{i}{\hbar}\tilde{S}(x, x_0; \Delta t)\right) + O(\Delta t^2) \quad (3.14)$$

where

$$\rho(x, x_0; \Delta t) = -\frac{\partial^2 \tilde{S}(x, x_0; \Delta t)}{\partial x_j \partial x_0}.$$

It turns out that this formula can be somewhat improved. The Van Vleck density $\rho(x, x_0; \Delta t)$ is explicitly given, taking formula (3.13) into account, by

$$\rho(x, x_0; \Delta t) = -\frac{m}{\Delta t} - \tilde{U}''_{x, x_0}(x, x_0)\Delta t$$

where

$$\tilde{U}''_{x, x_0} = -\frac{\partial^2 \tilde{U}(x, x_0)}{\partial x_j \partial x_k}.$$

Writing

$$\begin{aligned} \left(-\frac{m}{\Delta t} - \tilde{U}''_{x, x_0}(x, x_0)\Delta t\right) &= -\frac{m}{\Delta t}[1 - m^{-1}\tilde{U}''_{x, x_0}(x, x_0)\Delta t^2] \\ &= -\frac{m}{\Delta t}[1 + O(\Delta t^2)], \end{aligned}$$

we thus have

$$\rho(x, x_0; \Delta t) = \frac{m}{\Delta t} (1 + O(\Delta t^2)). \quad (3.15)$$

Writing

$$\tilde{\rho}(\Delta t) = \frac{m}{\Delta t} \quad (3.16)$$

which is just the Van Vleck density for the free particle Hamiltonian, we have

$$\rho(x, x_0; \Delta t) = \tilde{\rho}(\Delta t)(1 + O(\Delta t^2)) \quad (3.17)$$

and hence we can rewrite formula (3.14) as

$$K(x, x_0; \Delta t) = \left(\frac{1}{2\pi i\hbar}\right)^{1/2} \sqrt{\tilde{\rho}(\Delta t)} \exp\left(\frac{i}{\hbar}\tilde{S}(x, x_0; \Delta t)\right) + O(\Delta t^2). \quad (3.18)$$

We will see below that this formula allows an easy study of the quantum potential for K .

4. Short-time Bohmian trajectories

Let us determine the quantum potential Q corresponding to the propagator $K = K(x, x_0; t)$ using the asymptotic formulas above. Recall that it describes an isotropic source of point-like particles emanating from the point x_0 at initial time $t_0 = 0$. We assume for notational simplicity that the number of degrees of freedom is $n = 1$ (for the general case see [16]). We have, by definition,

$$Q = -\frac{\hbar^2}{2m\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2}$$

which we can rewrite

$$Q = -\frac{\hbar^2}{2m} \frac{\nabla_x^2 \sqrt{\rho}}{\sqrt{\rho}}.$$

We have, using (3.17),

$$\sqrt{\rho} = \sqrt{\tilde{\rho}(\Delta t)}(1 + O(\Delta t^2))$$

and hence

$$\frac{\partial^2 \sqrt{\rho}}{\partial x^2} = O((\Delta t)^2).$$

From this it follows that the quantum potential associated with the propagator satisfies

$$Q(x, x_0; \Delta t) = O(\Delta t^2). \quad (4.1)$$

This suggests that the quantum trajectory of a sharply located particle should be identical with the classical (Hamiltonian) trajectory for short times. Let us show that this is, indeed, the case. If we want to monitor the motion of such a particle, we have of course to specify its initial momentum which gives its direction of propagation at time $t_0 = 0$; we set

$$p(0) = p_0. \quad (4.2)$$

In view of formula (2.1), the trajectory in position space is obtained by solving the system of differential equations

$$\dot{x} = \frac{\hbar}{m} \operatorname{Im} \left(\frac{1}{K} \frac{\partial K}{\partial x} \right), \quad x(0) = x_0. \quad (4.3)$$

Replacing the propagator K with its approximation

$$\tilde{K}(x, x_0; \Delta t) = \left(\frac{1}{2\pi i \hbar} \right)^{1/2} \sqrt{\tilde{\rho}(\Delta t)} \exp \left(\frac{i}{\hbar} \tilde{S}(x, x_0; \Delta t) \right)$$

we have, since $K - \tilde{K} = O(\Delta t^2)$ in view of (3.18),

$$\dot{x} = \frac{\hbar}{m} \operatorname{Im} \left(\frac{1}{\tilde{K}} \frac{\partial \tilde{K}}{\partial x} \right) + O(\Delta t^2).$$

A straightforward calculation, using the expression (3.13) for the approximate action $\tilde{S}(x, x_0; \Delta t)$, leads to the equation

$$\dot{x}(\Delta t) = \frac{x(\Delta t) - x_0}{\Delta t} - \frac{1}{m} \frac{\partial \tilde{U}}{\partial x}(x(\Delta t), x_0) \Delta t + O(\Delta t^2) \quad (4.4)$$

(*cf.* the proof of Lemma 248 in [14]). This equation is singular at time $t = 0$ hence the initial condition $x(0) = x_0$ is not sufficient for finding a unique solution; this is of course consistent

with the fact that (4.4) describes an arbitrary particle emanating from x_0 ; to single out one quantum trajectory we have to use the additional condition (4.2) giving the direction of the particle at time $t = 0$ (see the discussion in Holland [22], §6.9). We thus have

$$x(\Delta t) = x_0 + \frac{p_0}{m}\Delta t + O(\Delta t^2);$$

in particular $x(\Delta t) = x_0 + O(\Delta t)$ and hence, by continuity,

$$\frac{\partial \tilde{U}}{\partial x}(x(\Delta t), x_0) = \frac{\partial \tilde{U}}{\partial x}(x_0, x_0) + O(\Delta t).$$

Let us calculate $(\partial \tilde{U} / \partial x)(x_0, x_0)$. We have, taking definition (3.12) into account,

$$\frac{\partial \tilde{U}}{\partial x}(x, x_0) = \int_0^1 \lambda \frac{\partial U}{\partial x}(\lambda x + (1 - \lambda)x_0, 0) d\lambda$$

and hence

$$\frac{\partial \tilde{U}}{\partial x}(x_0, x_0) = \int_0^1 \lambda \frac{\partial U}{\partial x}(x_0, 0) d\lambda = \frac{1}{2} \frac{\partial U}{\partial x}(x_0, 0).$$

We can thus rewrite equation (4.4) as

$$\dot{x}(\Delta t) = \frac{x(\Delta t) - x_0}{\Delta t} - \frac{1}{2m} \frac{\partial U}{\partial x}(x_0, 0) \Delta t + O(\Delta t^2).$$

Let us now differentiate both sides of this equation with respect to Δt :

$$\ddot{x}(t) = \frac{\dot{x}(t) - \dot{x}_0}{(\Delta t)^2} + \frac{\dot{x}(t)}{\Delta t} - \frac{1}{2m} \frac{\partial U}{\partial x}(x_0, 0) + O(\Delta t) \quad (4.5)$$

that is, replacing $\dot{x}(\Delta t)$ by the value given by (4.4),

$$\dot{p}(t) = m\ddot{x}(t) = -\frac{\partial U}{\partial x}(x_0, 0) + O(\Delta t). \quad (4.6)$$

Solving this equation we get

$$p(t) = p_0 - \frac{\partial U}{\partial x}(x_0, 0) \Delta t + O(\Delta t^2). \quad (4.7)$$

Summarizing, the solutions of the Hamilton equations are given by

$$x(\Delta t) = x_0 + \frac{p_0}{m} \Delta t + O(\Delta t^2) \quad (4.8)$$

$$p(\Delta t) = p_0 - \frac{\partial U}{\partial x}(x_0, 0) \Delta t + O(\Delta t^2). \quad (4.9)$$

These equations are, up to the error terms $O(\Delta t^2)$ the equations of motion of a classical particle moving under the influence of the potential U ; there is no trace of the quantum potential, which is being absorbed by the terms $O(\Delta t^2)$. The motion is thus identical with the classical motion on time scales of order $O(\Delta t^2)$

4.1. The Zeno effect in the Bohm approach

Now let us consider how Zeno's paradox emerges from the Bohm approach as outlined in the previous section. Consider the transition process $|1\rangle \leftrightarrow |2\rangle$ to illustrate the principle involved. This electric dipole transition is driven by an electric field $\mathbf{E} = \mathbf{E}_1 \cos \omega t$ so that

$$|\psi_{12}(t)\rangle = a(t)|1\rangle + b(t)|2\rangle. \quad (4.10)$$

Since $|a|^2 + |b|^2 = 1$, we need only calculate $a(t)$ using the Schrödinger equation $i\hbar da/dt = Ha$. Here $H = -\Omega = \mu|E_1|$ for the rotating-wave model [35].

In order to use the Bohm model on this transition², we must rewrite equation (4.10) in the form

$$\Psi(\mathbf{x}, t) = a(t)\psi_1(\mathbf{x}, t) + b(t)\psi_2(\mathbf{x}, t) = R^\Psi(\mathbf{x}, t)e^{\frac{i}{\hbar}S^\Psi(\mathbf{x}, t)}$$

so that the momentum of the electron undergoing the transition is

$$p^\Psi(\mathbf{x}, t) = \nabla S^\Psi(\mathbf{x}, t). \quad (4.11)$$

while the quantum potential driving the transition is

$$Q_{12}^\Psi(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \frac{\nabla^2 R^\Psi(\mathbf{x}, t)}{R^\Psi(\mathbf{x}, t)}. \quad (4.12)$$

When $a(t) > 0 \neq 1$ equation (4.11) shows there is a non-zero current between $|1\rangle$ and $|2\rangle$, which means that the electron is moving from level 1 to level 2 or vice versa depending on the sign of the current. Equation (4.12) gives the quantum energy driving the transition. If $a(t) = 0$, the electron is in the state $\psi_2(\mathbf{x}, t)$. Equation (4.11) then shows its momentum is the momentum it would have in state $|2\rangle$, while equation (4.12) gives the value of its quantum potential in that state. There is no longer any quantum energy driving a transition so the electron stays in that level until a new perturbation again forces a transition. What this means is that if we can prevent the quantum potential Q_{12}^Ψ from developing, the electron in state 2 becomes trapped there.

Thus the quantum potential plays an essential role in the 'watched pot' effect. In order to prevent the transition taking place we must introduce a suitable *physical process* which prevents this quantum potential from developing. Thus the intervention is not simply 'watching'. We need a process that actually interacts with the electron. This process must be repudiative and quick enough to prevent the quantum potential developing. This is the role played by the continuous wave optical pulses in the experiment of Itano *et al* [25]. This is an example of the participatory nature of a 'measurement' proposed by Wheeler [37] and by Bohm and Hiley [5]. However notice it is not simply the process of 'watching'.

5. Conclusion

In this paper we have shown how the question raised by Misra and Sudarshan [32] in their discussion of α -decay can be answered using the Bohm-Hiley approach [5]. In order to inhibit α -decay occurring, we must use a process that directly interacts with the unstable nuclei. It is not sufficient to surround the nuclei with a passive detection device like a bubble or cloud chamber. Merely detecting the α -particle outside the nucleus is not sufficient to inhibit the decay.

We have shown that Bohm's dynamics resides in the notion of a Hamiltonian flow. In such an approach, both the classical and quantum behaviours emerge from the same mathematical

² We will only present a very simplified discussion of this transition here. A full detailed account of such a transition has been given by Lam and Dewdney [28].

structure, the symplectic structure. It is the appearance of the quantum potential, Q^Ψ , in equation (1.1) that differentiates the two types of mechanics. When Q^Ψ is negligible compared with the kinetic energy, the equation simply reduces to the classical Hamilton-Jacobi equation and classical mechanics emerges. Hiley and Aziz Mufti [19] have illustrated how this occurs in a specific example based on an ‘upside-down’ harmonic potential, a potential that was used to illustrate the ideas behind the inflationary cosmological model of Guth and Pi [18].

At first sight it might seem rather ironic that the suppression of the quantum potential leads to classical behaviour and at the same time prevents atomic transitions. However it must be remembered that quantum mechanics was introduced to explain the stability of matter and the Bohm approach shows that the quantum potential plays a key role in stationary states. Indeed if one examines the ground states of atoms and nuclei, we find that the quantum potential plays a key role in determining their energy values. In the ground state of the Schrödinger hydrogen atom, for example, the energy eigenvalue *is all* quantum potential energy.

Thus in order to make a transition between energy levels, a change in the quantum potential energy is essential. Then clearly if we want to prevent such a transition from occurring, we need to stop the quantum potential energy from changing. To achieve this, we need some actual physical process to interact with the system. This then is the explanation of the quantum Zeno effect in the Bohmian approach.

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