Spin and charge from space and time

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Spin and charge from space and time

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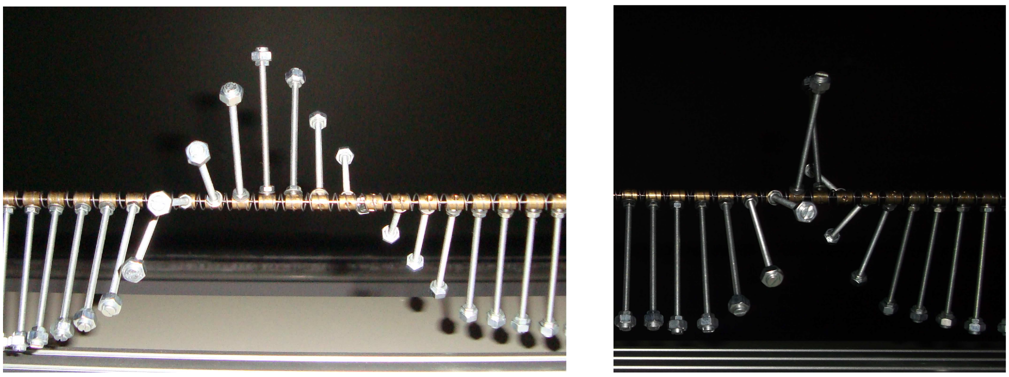
Abstract. We follow the idea that particles are topological solitons, characterised by topological quantum numbers determining charge and spin. To describe such particles and their spin we use the three rotational degrees of freedom of local spatial dreibeins. Orbiting particles contribute by internal rotation to the total angular momentum. The mass of these particles is given by their field energy only.

1. Introduction

A pendulum model as depicted in Fig. 1 obeys a discretised version of the Sine–Gordon model for the rotational angle ϑ, a bosonic field in one time and and one space dimension, 1+1D [1]. Besides wave-like excitations this interesting model has stable localised excitations, kink solutions, solitons, behaving like particles. Their mass is field energy only. The wave-velocity c in the mechanical model is of the order of 1m/s. The Sine-Gordon model is a relativistic model. Therefore, solitons moving with velocity v are Lorentz contracted, see Fig. 1b, and their mass increases with the well–known γ factor, $\gamma = 1/\sqrt{1 - v^2/c^2}$, $m(v) = \gamma m(0)$.



(a) (b)

Figure 1. A pendulum model obeying a discretised version of the Sine–Gordon model. A soliton at rest (a) has larger size than a moving soliton (b) which is Lorentz contracted. Mechanical model by Peter Pataki, photos by Gerald Pechoc.

There are kinks and antikinks, right-handed and left-handed solitons where the bosonic field ϑ varies from 0 to $\pm 2\pi$. Two–soliton configurations are time dependent since equal "charges" repel and opposite charges attract each other. Despite the fact that solitons have a finite size, in collisions they behave point-like. The corresponding time-depend two-soliton solutions are well known [1]

$$\vartheta(x,t) = 4\,\mathrm{atan}\frac{\beta\sinh(\gamma x)}{\cosh(\beta\gamma t)}. \tag{1}$$

In Fig. 2 such solutions are depicted for some values of $\beta = v/c$, for $t = 0$, the time of shortest approach. Towards $\beta = 1$ the common profile gets steeper and steeper and converges to size zero, a step function.

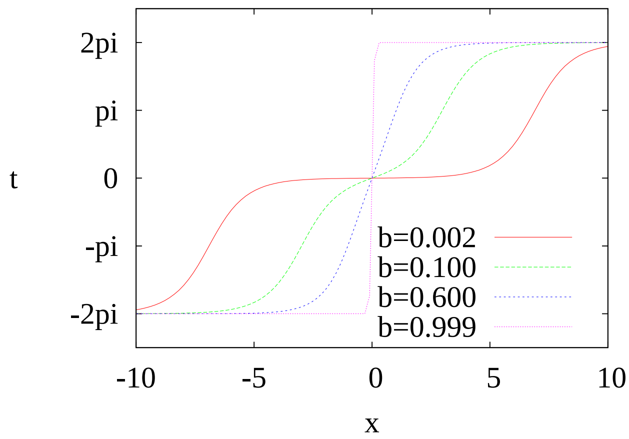


Figure 2. Field of two colliding solitons at $t = 0$, the time of shortest approach. For large values of β the two solitons are not really distinguishable. Approaching $\beta = 1$ the common profile gets arbitrary steep, closer and closer to a step function.

A generalisation of the 1+1D Sine–Gordon model to 3+1D was suggested in ref. [2], the model of topological fermions (MTF). There the bosonic variable ϑ of the Sine–Gordon model is generalised to three Euler angles, to an SO(3) field. In this article we interpret this field of rotational matrices as orientations of local dreibeins in space which may rotate in time. As in the Sine–Gordon model the MTF has stable topological excitations and mass is field energy. Despite the fact that these solitons are extended, in high energy collisions they are compressed to nearly point-like particles. Charge and spin are topological quantum numbers.

Let us recall the most important properties of charge and spin. Charges feel Coulomb and Lorentz forces. Since Faraday's laws of electrolysis, J. J. Thomsons cathode ray experiments in 1897 and Millikan's oil drop experiments in 1909 we know that charge is quantised in units of the elementary electric charge $e_0 = 1.602 \cdot 10^{-19}$ C. Maxwell's electrodynamics and quantum mechanics give no explanation for this quantisation of electric charge.

There is no classical analogon of spin. From quantum mechanics we know that electrons have two spin-states, usually indicated as spin up and spin down. The spin is described by the SU(2) group, like the isospin. Whereas the isospin is an internal degree of freedom the spin has a relation to our 3–dimensional space: the spin $\mathbf{s}$ contributes like the orbital angular momentum $\mathbf{l}$ to the angular momentum $\mathbf{j}$ of particles, $\mathbf{j} = \mathbf{l} + \mathbf{s}$.

From special relativity we know that there is no absolute space and space and time are related. Gravitation theory teaches us that space–time is not rigid, it is deformable by matter and energy density. The length scales of space–time depend on the position, they are encoded in the metric tensor $g_{\mu\nu}$ in gravitational theory and described by local translations in Poincarè gauge theory.

2. The model

Now we follow the idea that charge is a property of space, a space which is Minkowskian at macroscopic distances and has rotational dislocations at microscopic distances as schematically shown in Fig. 3. We will show that dislocations which are described by a dreibein (triade) rotating in space by 2π with an appropriate Lagrangian is a nice model to describe elementary charges.

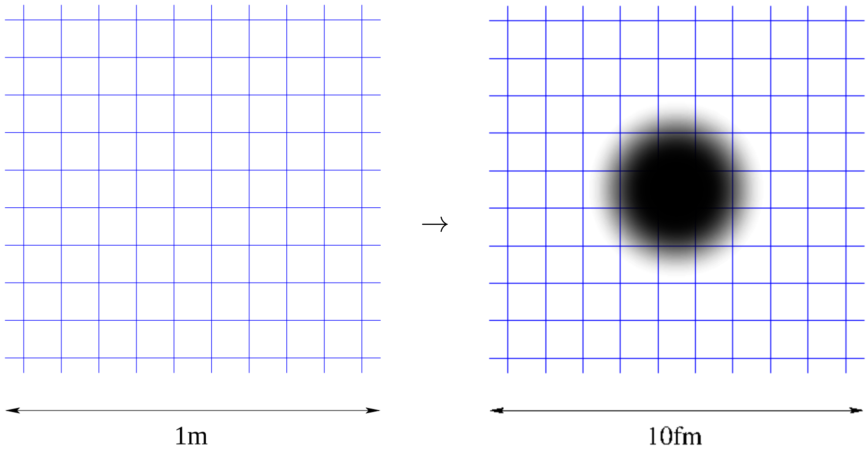


Figure 3. Schematic picture of space which behaves Minkowskian at macroscopic distances and is perturbed by dislocations at microscopic distances.

2.1. Degrees of freedom

The only degrees of freedom of this model are the above mentioned three rotational angles of spatial dreibeins (triades). The earliest description of rotations is due to Rodrigues (1840) by unit quaternions [3]

$$Q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}: \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \quad q_0^2 + \vec{q}^2 = 1, \ \vec{q} = (q_1, q_2, q_3). \tag{2}$$

and was reinvented by Hamilton (1843). Today it is more common to use the group SO(3) or its double covering group SU(2) which is isomorphic to S^3, the sphere in 4D. We make a transition from the three imaginary quaternionic units to Pauli matrices

$$(\mathbf{i}, \mathbf{j}, \mathbf{k}) = -\mathrm{i}\vec{\sigma}, \quad x^\mu = (ct, \mathbf{r}), \quad q_0 = \cos\alpha, \ \vec{q} = \vec{n}\sin\alpha, \ \vec{n}^2 = 1, \tag{3}$$

and describe the dreibein field by an SU(2) field

$$Q(x) = \cos\alpha(x) - \mathrm{i}\vec{\sigma}\vec{n}(x)\sin\alpha(x) = \mathrm{e}^{-\mathrm{i}\alpha(x)\vec{\sigma}\vec{n}(x)}. \tag{4}$$

Models with such degrees of freedom are usually called non-linear sigma models. We have to keep in mind that due to the double covering, the SO(3) field configuration corresponding to $\pm Q(x)$ are identical.

2.2. Geometry of S^3

By the SU(2) field $Q(x)$, curves in space-time, parametrised by s or t: $x^\mu(s), x^\mu(t)$ are mapped to SU(2) and its parameter space, the sphere S^3, and reflect its geometry. To every point of such curves we can attribute a vector field, the connection one-form,

$$(\partial_s Q)\, Q^\dagger = -\mathrm{i}\vec{\sigma}\vec{\Gamma}_s \tag{5}$$

vectors with values in the tangential space of S^3. With a few lines one can see that in the parametrisation (4) the three-component vector $\vec{\Gamma}_s$ reads

$$\vec{\Gamma}_s = \partial_s\alpha\vec{n} + \sin\alpha\cos\alpha\,\partial_s\vec{n} + \sin^2\alpha\,\vec{n}\times\partial_s\vec{n}. \tag{6}$$

The arrows are used to avoid too many indices and indicate the vector character in the tangential space.

Further we can attribute to every area in space-time, e.g. $\mathrm{d}x\mathrm{d}y$, an area on S^3 and get an area density, a connection two-form or curvature by

$$\vec{R}_{\mu\nu} = \vec{\Gamma}_\mu \times \vec{\Gamma}_\nu, \tag{7}$$

as schematically shown in Fig. 4. Due to the equality of mixed partials, applied to Eq. (5), we

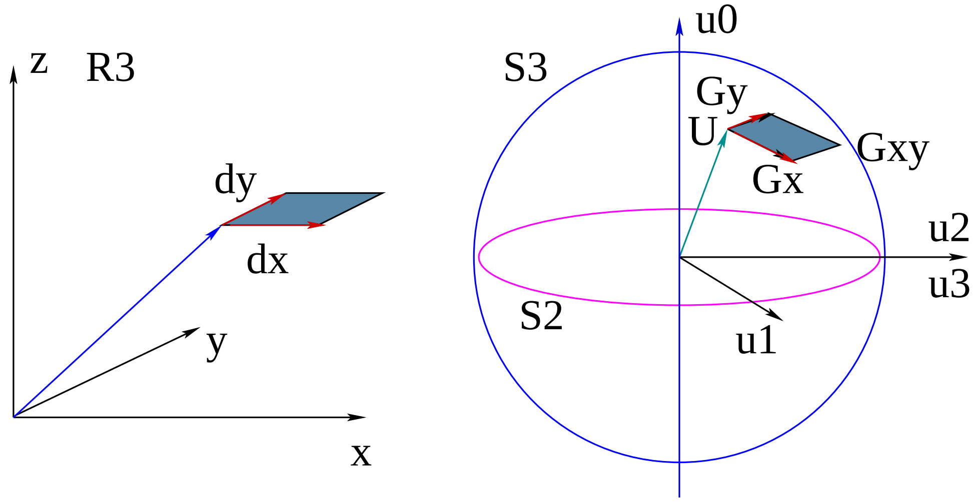


Figure 4. Schematic picture of maps of points, line and area elements from Minkowski space to S^3 : $\mathrm{d}x\wedge\mathrm{d}y \mapsto \vec{\Gamma}_x\times\vec{\Gamma}_y\mathrm{d}x\mathrm{d}y$

get the Maurer-Cartan structural equation

$$\vec{\Gamma}_\mu \times \vec{\Gamma}_\nu = \frac{1}{2}[\partial_\mu\vec{\Gamma}_\nu - \partial_\nu\vec{\Gamma}_\mu] \tag{8}$$

leading to the well-known curvature expression

$$\vec{R}_{\mu\nu} = \partial_\mu\vec{\Gamma}_\nu - \partial_\nu\vec{\Gamma}_\mu - \vec{\Gamma}_\mu\times\vec{\Gamma}_\nu, \tag{9}$$

which is covariant against basis rotations.

2.3. Relate geometry to physics

Defining the field strength tensor we start to relate the geometry to physics. With units appropriate to get Gauß's law for the elementary electric charge e_0 the dual field strength tensor reads

$$^\star\vec{F}_{\mu\nu} = \frac{e_0}{4\pi\epsilon_0}\vec{R}_{\mu\nu}. \tag{10}$$

Electric fields are described by spatial components of the curvature tensor and magnetic fields by space-time components. The arrow in $^\star\vec{F}_{\mu\nu}$ indicates that this model has three contributions for every field strength component. Like in QCD we call these internal components “color”. Correspondingly we can introduce a dual vector field in appropriate units

$$\vec{C}_\mu = -\frac{e_0}{4\pi\varepsilon_0 c}\vec{\Gamma}_\mu. \tag{11}$$

2.4. Lagrangian

In relativistic field theories the most effective method to define the dynamics is by defining a Lagrangian. Our aim is to describe charges and their interaction with the electro-magnetic field. There is a most successful theory for this aim and this is Maxwell's electrodynamics. But it describes charges and their fields by different quantities, by fermion fields $\psi(x)$ and by gauge fields $A^\mu(x)$. From the successes of the Sine-Gordon model we get the idea to describe charges and their fields on the same footing, by a single field $Q(x)$. We aim to investigate how far we can get with this idea. We expect a non-linear theory. Maxwell's electrodynamics should be a very clever linearisation of this model by the above mentioned separation of charges and field.

From the success of gauge field theories we expect a Lagrangian quadratic in the field strength tensor. From the comparison with the Sine-Gordon model we know that stable solitonic configurations of finite energy can only be obtained with two competing contributions to the action, one which tries to smooth solitons and one shrinking solitons. Extended solitons, reacting at large energies like point-like particles need to be regular solutions without any substructure. Such a field configuration symmetric in three-dimensional space is a configuration of hedge-hog type. It should be convenient for point-like charges.

The only known possibility to get non-trivial stable field configurations with long-range interactions is by the Lagrangian

$$\mathcal{L} = \mathcal{L}_e - \mathcal{H}_p = -\frac{\alpha_f \hbar c}{4\pi}\left(\frac{1}{4}\vec{R}_{\mu\nu}\vec{R}^{\mu\nu} + \Lambda\right), \quad \frac{\alpha_f \hbar c}{4\pi}\frac{1}{4}\vec{R}_{\mu\nu}\vec{R}^{\mu\nu} = -\frac{1}{4\mu_0}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu}. \tag{12}$$

The prefactor $\frac{\alpha_f \hbar c}{4\pi}$ follows from the expression (10) for the field strength tensor, it is necessary to get the dimension of an action density in the International System of Units (SI). The term quadratic in the field strength is a kinetic term describing in electrodynamics the dynamics of the photon fields. It agrees with the Skyrme term of the Skyrme model [4]. It contains four derivatives of the soliton field $Q(x)$, maximally two of them are time derivatives. The Skyrme model includes another term, quadratic in derivatives, corresponding to the usual kinetic term for the connection field $\vec{\Gamma}_\mu$. As the Skyrme model shows, this term leads also to solitonic solutions, but to solitons with short range interactions. The Skyrme model is therefore accepted as an approximation to QCD. As we want to describe long range Coulombic interactions we have to avoid this kinetic term. According to the Hobart-Derrick theorem we have to include a term with less than three derivatives and the only possibility left is a term without derivatives, a potential term Λ as in Eq. (12). Solitons with properties of charges have fields extending up to infinity. At infinity the potential term should contribute with its vacuum value only. The only term left at infinity is therefore the kinetic term, quadratic in the field strengths. To get non-vanishing derivatives of fields and a constant potential term we have to have degenerate vacua. A good choice to allow for hedge-hog type solitons is a term $\Lambda(q_0)$ with even powers in q_0, schematically shown in Fig. 5, with potential minima at the equatorial sphere S^2_{equ},

$$\Lambda(q_0) = \frac{q_0^{2m}}{r_0^4},\ m = 1, 2, 3, \cdots . \tag{13}$$

By dimensional reasons, to equilibrate the four derivatives in $\vec{R}_{\mu\nu}\vec{R}^{\mu\nu}$, this term has to include the forth power of a scale parameter r_0 which finally determines the size of solitons.

Like the cosmological constant in general relativity the term (13) has no derivatives. But in distinction to the cosmological constant it is coordinate dependent, it contributes in the center of solitons and does not contribute in the vacuum. One could call $\Lambda(q_0)$ a cosmological function. If the universe would start in a state with $Q = 1$ and $\alpha = 0$ one would expect an inflationary transition to a vacuum state with $\alpha = \pi/2$.

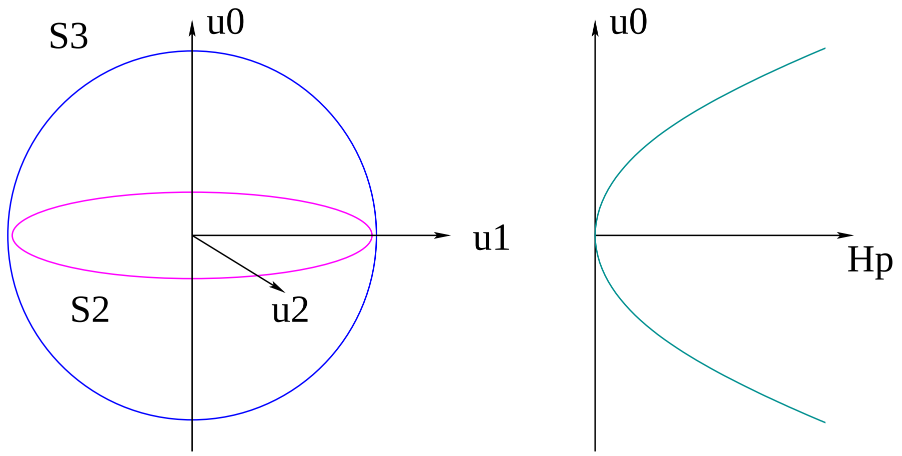


Figure 5. Schematical drawing of the potential term $\Lambda(q_0)$ with even powers in q_0 leading to a two-fold degeneracy of the vacuum states at the equatorial $S^2_{\rm equ}$.

3. Single solitons and charge quantisation

After defining the Lagrangian (12) and specifying the potential (13) we are ready to derive the equations of motion and to check whether there are long-range solitonic solutions. With an hedge-hog ansatz we construct spherical symmetric solutions,

$$\vec{n}(x) = \frac{\vec{r}}{r}, \quad \alpha(x) = \alpha(\rho), \quad \rho = r/r_0. \tag{14}$$

For vacuum configurations $\alpha = \pi/2$, Eq. (6) is reducing to the vector field of a dual Dirac monopole in the Wu-Yang representation with

$$\vec{\Gamma}_\mu = \vec{n} \times \partial_\mu \vec{n}. \tag{15}$$

The original Dirac monopole [5, 6] has two types of singularities, the famous Dirac-string and the singularity in the center. In the Wu-Yang representation of the Dirac monopole [7, 8] the Dirac string is removed, but the singularity in the center is still present. With the ansatz (14) we are able to remove also the singularity in the center.

Since there is no time dependence in the ansatz (14), we can use spherical coordinates r, ϑ, φ and get from Eq. (6) for the connection field

$$\vec{\Gamma}_r = \partial_\rho \alpha(\rho)\,\vec{n}, \;\; \vec{\Gamma}_\vartheta = \sin\alpha\left[\cos\alpha\,\vec{e}_\theta + \sin\alpha\,\vec{e}_\phi\right], \;\; \vec{\Gamma}_\varphi = \sin\theta\sin\alpha\left[\cos\alpha\,\vec{e}_\phi - \sin\alpha\,\vec{e}_\theta\right], \tag{16}$$

with $\vec{e}_\theta, \vec{e}_\phi$ unit vectors in internal color space. With the diagonal, spherical metric $g_{\mu\nu} = \mathrm{diag}(1, l_r^2, l_\vartheta^2, l_\varphi^2)$ and $(l_r, l_\vartheta, l_\varphi) = (1, r, r\sin\vartheta)$ we get the dual vector field

$$\vec{C}_r = -\frac{e_0}{4\pi\varepsilon_0 c}\frac{\vec{\Gamma}_r}{l_r}, \quad \vec{C}_\vartheta = -\frac{e_0}{4\pi\varepsilon_0 c}\frac{\vec{\Gamma}_\vartheta}{l_\vartheta}, \quad \vec{C}_\varphi = -\frac{e_0}{4\pi\varepsilon_0 c}\frac{\vec{\Gamma}_\varphi}{l_\varphi}. \tag{17}$$

The non-vanishing field strength components read

$$\begin{aligned}
\vec{E}_r &= -\frac{e_0}{4\pi\varepsilon_0}\frac{\vec{\Gamma}_\vartheta \times \vec{\Gamma}_\varphi}{r^2\sin\vartheta} = -\frac{e_0}{4\pi\varepsilon_0}\frac{\sin^2\alpha}{r^2}\vec{n}, \\
\vec{E}_\vartheta &= -\frac{e_0}{4\pi\varepsilon_0}\frac{\vec{\Gamma}_\varphi \times \vec{\Gamma}_r}{r\sin\vartheta} = -\frac{e_0}{4\pi\varepsilon_0}\frac{\alpha'(r)\,\sin\alpha}{r}(\cos\alpha\,\vec{e}_\theta + \sin\alpha\,\vec{e}_\phi), \\
\vec{E}_\varphi &= -\frac{e_0}{4\pi\varepsilon_0}\frac{\vec{\Gamma}_r \times \vec{\Gamma}_\vartheta}{r} = -\frac{e_0}{4\pi\varepsilon_0}\frac{\alpha'(r)\,\sin\alpha}{r}(\cos\alpha\,\vec{e}_\phi - \sin\alpha\,\vec{e}_\theta).
\end{aligned} \tag{18}$$

From the Lagrangian (12) follows the energy functional

$$H = H_e + H_p = \frac{\alpha_f \hbar c}{r_0} \int_0^\infty \mathrm{d}\rho \left[\frac{\sin^4 \alpha}{2\rho^2} + (\partial_\rho \cos\alpha)^2 + \rho^2 \cos^{2m} \alpha \right] \tag{19}$$

and by its variation the non-linear differential equation

$$\partial_\rho^2 \cos\alpha + \frac{(1-\cos^2\alpha)\cos\alpha}{\rho^2} - m\rho^2 \cos^{2m-1}\alpha = 0 \tag{20}$$

for $\cos\alpha(\rho)$. For this equation analytic solutions are known for $m = 2$ and $m = 3$ [2, 9]. From an aesthetic point of view the $m = 3$ solution looks nicer, it reads

$$\alpha(\rho) = \arctan\rho \quad \text{with} \quad \rho = \frac{r}{r_0}. \tag{21}$$

The radial energy density, the integrand in the energy functional (19), has three contributions,

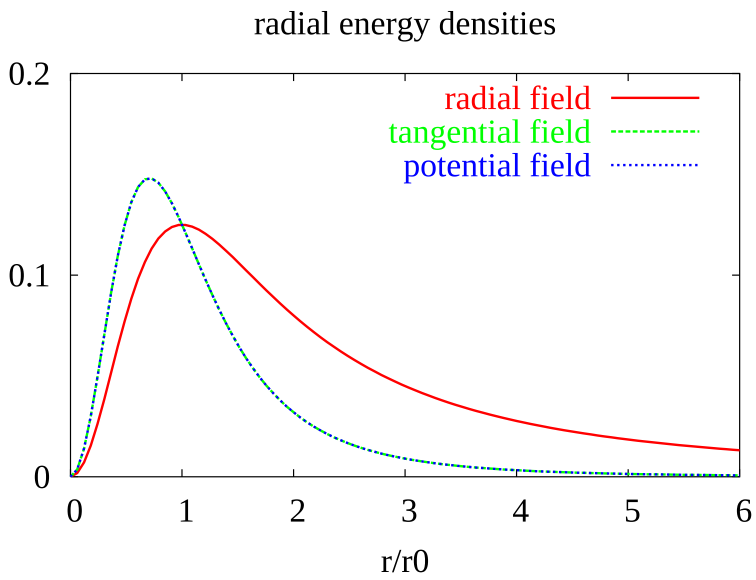


Figure 6. Radial energy densities (22) for the solution (21) of the differential equation (20) for $m = 3$.

$$h = \frac{\alpha_f \hbar c}{r_0} \Big[\frac{\rho^2}{2(1+\rho^2)^2} + \frac{\rho^2}{(1+\rho^2)^3} + \frac{\rho^2}{(1+\rho^2)^3} \Big]. \tag{22}$$

Up to the constant factor in front of the bracket in Eq. (22) they are shown in Fig. (6). For large distances the radial field decays with $1/r^4$ as expected for the energy density of the electric field of a point charge. Due to the regularity of the solution the singularity of the electric field of a point-like charge is removed. There is no singularity left. Two further terms contribute to the energy density, a tangential term originating in the components $\vec{E}_\vartheta$ and $\vec{E}_\varphi$ of the electric field and a contribution from the potential energy H_p. For the solution (21) the latter two contributions agree.

The total energy of this single soliton solution reads

$$E_1 = \frac{\alpha_f \hbar c}{r_0}\frac{\pi}{4} \quad \text{with} \quad \alpha_f \hbar c = 1.44 \text{ MeV fm}. \tag{23}$$

Comparing this result with the energy of the lightest charge particle, the electron with a rest energy of 0.511 MeV, we get $r_0 = 2.21$ fm. This value is of the same order of magnitude as the classical electron radius $\frac{\alpha_f \hbar c}{m_e c^2} = \alpha_f \frac{\hbar}{m_e c} = 2.82$ fm.

In the SI the scales for space, time, mass and charge are based on the requirement for a coherent system of units. In a certain sense these scales are man-made and can't be predicted by a model, they can only be adjusted by a comparison to experimental values. The only parameters of the model, r_0, c, $\hbar$ and α_f, correspond to these four scales. They were adjusted to the charge and the mass of the electron, the velocity of light and the dielectric constant of the vacuum. Therefore, the elementary electric charge can't be predicted. But, in distinction to Maxwell's electrodynamics this model predicts the quantisation of electric charge. Due to the geometric restrictions fractional multiples of e_0 can't be realised by regular field distribution of the SU(2)-field $Q(x)$. There is no contradiction to the existence of quarks, since they never appear as isolated particles.

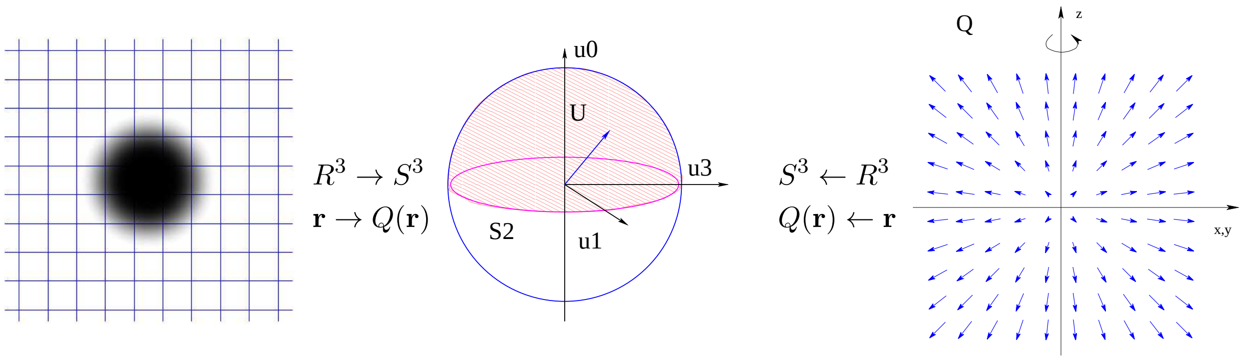


Figure 7. The field of rotational matrices $Q(\mathbf{r})$ of spatial dreibeins maps $R^3 \to S^3$. The center of the soliton is mapped to the origin and spatial infinity to the equatorial $\mathrm{S}^2_{\mathrm{equ}}$. The right diagram indicates this map by arrows of length $\sin\alpha(\mathbf{r})$ and direction $\vec{n}(\mathbf{r})$. A soliton covers only half of S^3. Points on the lower half of S^3 are furthermore characterised by dashed arrows.

We succeeded to follow the idea of Fig. 3 to describe a rotational dislocation in R^3 by assigning to every coordinate triple $\mathbf{r}$ a rotational matrix $Q(\mathbf{r})$, as shown in Fig. 7. A unit charge covers a hemisphere of S^3 only. To get a more precise picture of the field, $Q(\mathbf{r})$ is symbolised in the right diagram by arrows of length $\vec{q}(\mathbf{r}) = \sin\alpha(\mathbf{r})\vec{n}(\mathbf{r})$. The center of the soliton is mapped to the unit matrix and therefore a vector of length zero. Traversing the center of the soliton along a line through the origin, $Q(\mathbf{r})$ defines a continuous 2π rotation with positive chirality. The rotational axis agrees with the direction of the line. In this sense the configuration is a three-dimensional generalisation of the Sine-Gordon model with solitons defined by 2π rotations between degenerate ground states. The ground states build a discrete set in the Sine-Gordon model and a two-dimensional manifold in this model.

Releasing the condition $\vec{n}(x) = \frac{\vec{r}}{r}$ and allowing for global rotations of soliton configurations we can realise a continuum of soliton solutions for the Lagrangian (12). All these configurations belong to the same homotopy class, they can be "continuously deformed" into each other. There are four homotopy classes of SU(2)-field configurations. They differ by transformations with the non-trivial center element $z = -\mathbb{1}_2$ of SU(2), $Q \to -Q$, or by parity transformation Π_n in internal space, $\vec{n} \to -\vec{n}$. The far field of a soliton determines its interaction with other solitons

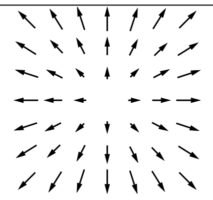
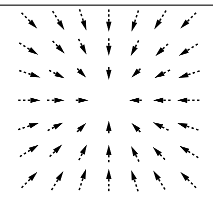
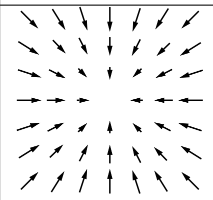
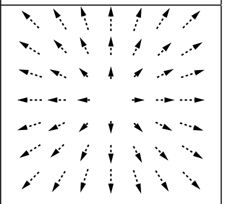
Transf.	1	z	Π_n	$z\Pi_n$
$\vec{n}$	$\vec{r}/r$	$-\vec{r}/r$	-$\vec{r}/r$	$\vec{r}/r$
q_0	≥ 0	≤ 0	≥ 0	≤ 0
Z	1	-1	-1	1
$2\mathcal{Q}$	1	1	-1	-1
diagram				

Table 1. There are four homotopy classes for soliton configurations. They differ by parity transformations in colour space $\Pi_n : \vec{n} \to -\vec{n}$ and transformations with the non-trivial center element $z = -\mathbb{1}_2 : Q \to zQ$. They can be characterised by the charge number Z and the topological charge $\mathcal{Q}$, defined in Eqs. (24) and (25). Length and direction of arrows depict the value of the imaginary part $\vec{q}(x) = \vec{n} \sin\alpha$ of the soliton field $Q(x)$. The sign of q_0 is indicated by the line type of the arrow, full lines for $q_0 > 0$ and dashed lines for $q_0 < 0$.

and therefore its charge. For the far field we have $\cos\alpha = 0$, arrows have unit length and arrows with full and dashed stem symbolise the same SU(2)-element on the equatorial $\mathrm{S}^2_{\mathrm{equ}}$. There are two possible coverings of $\mathrm{S}^2_{\mathrm{equ}}$. As discussed in detail in refs. [2, 9, 10, 11] we can distinguish them by choosing a closed surface $\mathcal{S}$ parametrised by coordinate lines u and v around the soliton center and determining

$$Z(\mathcal{S}) := -\frac{1}{4\pi} \oint_{\mathcal{S}} \mathrm{d}u\mathrm{d}v\, [\partial_u \vec{n} \times \partial_v \vec{n}]\vec{n}. \tag{24}$$

$Z(\mathcal{S})$ is the charge number of the soliton. It is obvious that Π_n transformations change the sign of Z. Configurations which are topological equivalent to a configuration with arrows pointing outwards we have defined as the charge of an electron, i.e. a negative elementary charge, $Z = -1$, arrows pointing inwards as positive, $Z = +1$. Now we are going to discuss in more detail the spin properties which are related to the hemispheres of S^3.

4. Spin properties

4.1. Spin as topological quantum number

From quantum mechanics we know that spin-1/2 particles appear in two spin states, usually called spin-up and spin-down. Only one component of the spin-vector is measurable and this component can have two values only, $\pm\frac{\hbar}{2}$. By the interaction of a spin-1/2 particle with a magnetic field the expectation value of the spin can be rotated in arbitrary direction. If we would attribute in our model spin-up and spin-down states to the upper and lower hemisphere of S^3, this would contradict to the experiment, since field configurations on different hemispheres are topological different, there is no continuous transition between them. This is the reason why, as mentioned above, we have to attribute SO(3) and not SU(2) field configurations to the solitons. Due to the double covering of SO(3) by SU(2), in SO(3) there is no upper and lower hemisphere, there is no difference between $\pm Q(x)$ configurations. In this picture a single soliton in the universe does not have a spin. Only by the interaction of solitons, the spin direction appears. There are two possibilities to connect two spatially separated SO(3) configurations continuously, a trivial connection and a non-trivial connection, see Fig. (8). The trivial connection reads in the SU(2) language, that the two solitons occupy the same hemisphere of S^3 and the non-trivial configuration, that the two solitons occupy different hemispheres. Schematic arrow diagrams for these two configurations for opposite charges are depicted in Fig. 8. The number of coverings of

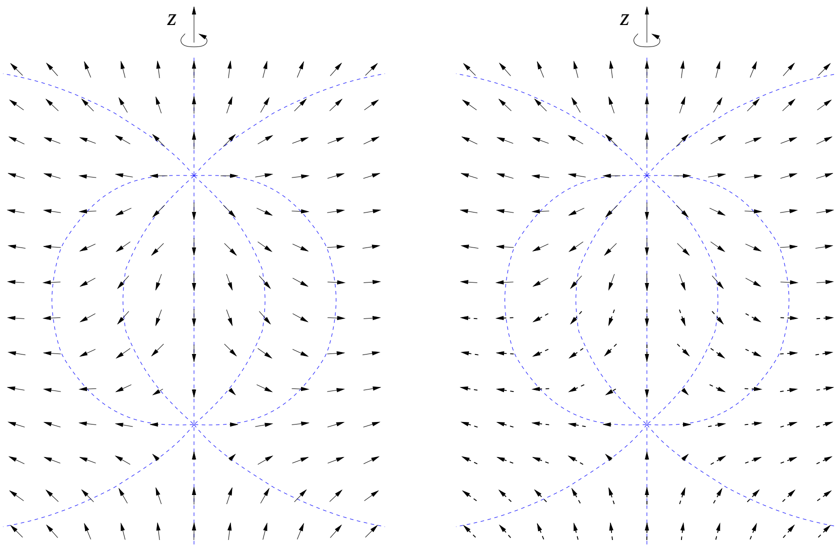


Figure 8. Schematic arrow diagram for the field $Q(\mathbf{r})$ and the electric field by blue (dashed) lines for two opposite unit charges. The arrows with full stem represent values $\vec{q} = \vec{n}\sin\alpha$ with positive values of $q_0 = \cos\alpha$, the dashed stems negative values of q_0, i.e. $\frac{\pi}{2} \leq \alpha \leq \pi$. The left configuration corresponds to a spin-zero and the right configuration to a spin-one state. The electric field lines follow lines of constant $\vec{n}$.

S^3 is a conserved topological quantum number. This is one of the reasons we try now to describe the spin quantum number s by this topological quantum number, the topological charge $\mathcal{Q}$,

$$\mathcal{Q} = \frac{1}{V(\mathbb{S}^3)} \int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, \vec{\Gamma}_r(\vec{\Gamma}_\vartheta \times \vec{\Gamma}_\varphi) \text{ with } V(\mathbb{S}^3) = \int_{S^2} \mathrm{d}^2 n \int_0^\pi \mathrm{d}\alpha \sin^2\alpha = 2\pi^2 \quad (25)$$

and define

$$s = |\mathcal{Q}|\,. \quad (26)$$

The field configurations of single solitons, see Table 1, cover half of S^3 and are therefore characterised by $s = 1/2$. Such configurations are connected to the surrounding by the lines of constant $\vec{n}$-field. A 4π-rotation of the center of a soliton can be described by a big circle on S^3. Since this lasso around S^3 can be contracted to a point, one can understand that by appropriate movements of the constant $\vec{n}$-lines the 4π-rotation is topological equivalent to no rotation. This reflects the famous 4π-rotational invariance of spin-1/2 states.

Looking more carefully to the attractive two-soliton configurations of Fig. 8, one can imagine that at spatial infinity all unit-arrows turn finally in positive z-direction for both configurations. This indicates that the configurations are uncharged, as mentioned above. The spin properties we can read off from the central structure of the configuration. We can attribute a chirality χ to the soliton centers according to the sign of the topological charge and get

$$\mathcal{Q} =: \chi s. \quad (27)$$

Especially significant is the Q-field along the z-axis. We realise that in the left diagram with $s = 0$ the soliton-field describes a left-handed rotation of the spatial dreibein for negative z-values and a right-handed rotation for positive z-values. These opposite chiralities of the two solitons we would describe in corresponding quantum mechanical configurations by opposite magnetic quantum numbers m_s. In the right diagram of Fig. 8 we observe a complete 4π-rotation along the z-axis as we expect for two parallel soliton spins adding up to $s = 1$. In the left, $s = 0$, diagram the soliton field $Q(\mathbf{r})$ at the center of mass $\mathbf{r} = 0$ of the two-soliton system approaches $\alpha = \pi/2$ for large distances of the solitons only. Therefore this configuration does not cover the full upper hemisphere twice. The energy of this configuration is thus lower than the energy of the $s = 1$-state in the left diagram where we get exactly $\alpha(\mathbf{r} = 0) = \pi/2$ and the configuration covers both hemispheres of S^3 completely.

4.2. Spin as angular momentum

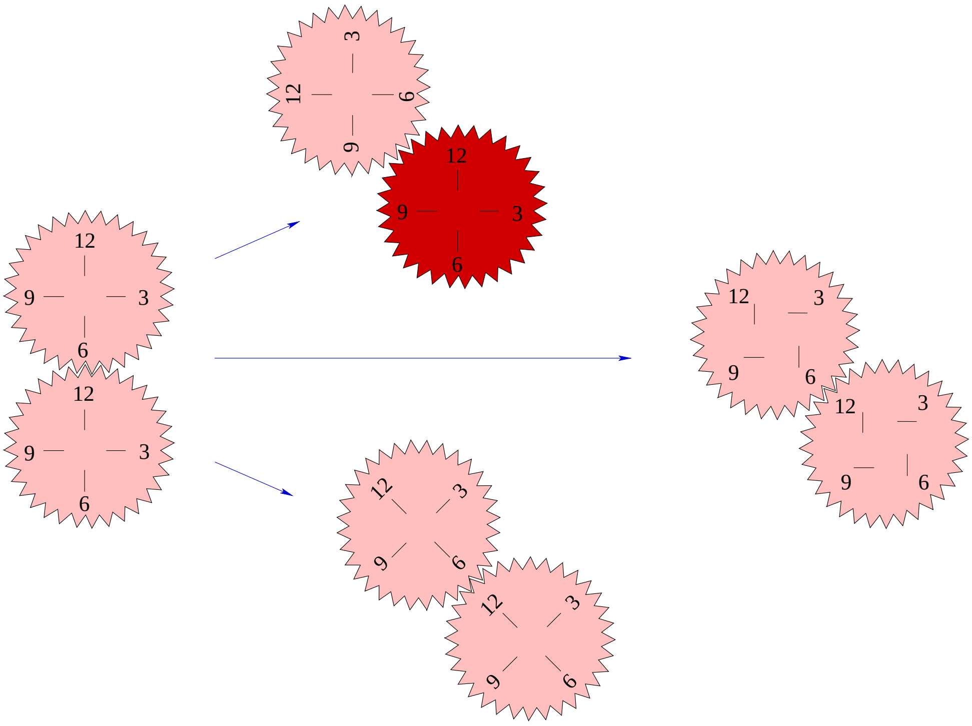


Figure 9. The rotation of gears of equal radius is compared to rotations of solitons. If a light gear rotates around a heavy (static) one, from the left to the upper diagram, the circulating gear performs a 4π-rotation. Another possibility for two gears of equal weight would be a rigid rotation, from the left to the lower diagram. Solitons circulate around as shown from left to right.

Whereas in quantum mechanics mainly the su(2)-algebra and the SU(2)-representations are important, the above discussion shows that in this model the manifold of the SU(2)-group and its map to R^3 is related to the spin properties. An essential property of spin we still didn't discuss, its contribution to angular momentum. In the static configurations of Fig. 8 nothing contributes to the angular momentum. But as soon as the solitons start to move around each other they have to rotate, they are connected to each other by the lines of constant $\vec{n}$-field, like gears. We can try to compare two unit charges to a pair of gears of equal size. If a heavy gear is at rest, another light gear circulating around performs a 4π-rotation, see Fig. 9 from the left to the upper diagram. For two gears of equal mass one would rather expect a rigid rotation, from left to bottom. But gears are rigid bodies and solitons are not rigid. That solitons perform a rigid rotation was a wrong assumption in ref. [9]. We are now going to discuss that solitons circulate rather as shown in Fig. 9 from left to right.

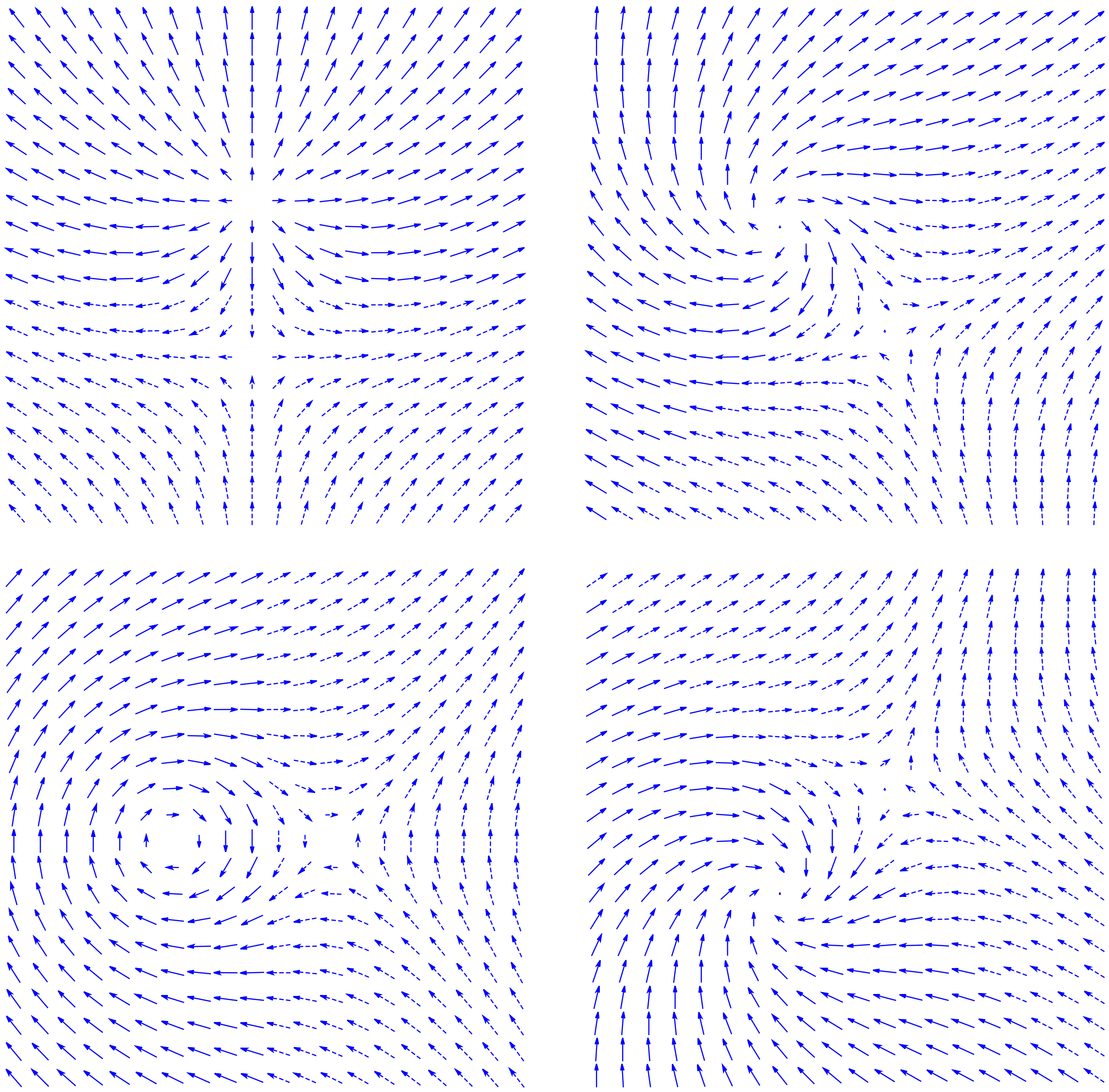


Figure 10. Two solitons with opposite charge and spin quantum number $s = 1$ rotating about a common center of mass. Four rotational angles $0, \frac{\pi}{4}, \frac{\pi}{2}$ and $\frac{3\pi}{4}$ are shown from left to right and top to bottom. This is not a rigid rotation. The line connecting the two soliton centers keeps its $\vec{n}$-value, $\vec{n} = (0, 0, -1)$ during the rotation.

For soliton configurations we have to take into account the boundary conditions defined by the broken vacuum. In Fig. 8 we assumed a vacuum configuration with $Q = -\mathrm{i}\vec{\sigma}\vec{n}$ and $\vec{n} = (0, 0, 1)$. This vacuum state has to remain unchanged during the rotation of the soliton pair. Therefore, solitons can't rotate like rigid bodies. In Fig. 10 a slow rotation of two solitons with opposite charges and $s = 1$ is shown for rotational angles of $0, \frac{\pi}{4}, \frac{\pi}{2}$ and $\frac{3\pi}{4}$. Along the line connecting the two soliton centers the vector $\vec{n}$ keeps its direction $\vec{n} = (0, 0, -1)$ during the rotation. It follows that there is an internal motion contributing to the total angular momentum. This two soliton state returns after a 2π-rotation to the original state as it should be for integer spin states.

4.3. Angular momentum components

We can treat $\alpha_k = \alpha n_k$ as field variables of the SU(2)-field $Q(x)$ in Eq. (4). The relations

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \alpha_k} \partial_\nu \alpha_k = \frac{\partial \mathcal{L}}{\partial \Gamma_{\mu k}} \Gamma_{\nu k} \tag{28}$$

allow to consider $\vec{\Gamma}_\mu$ as generalised velocities and to derive from the Lagrange density (12) generalised momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \vec{\Gamma}_\mu} = -\frac{\alpha_f \hbar c}{4\pi} \vec{\Gamma}_\nu \times \vec{R}^{\mu\nu}. \tag{29}$$

The canonical energy-momentum tensor

$$\Theta^\mu{}_\nu(x) = \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \alpha_k)} \partial_\nu \alpha_k - \mathcal{L}(x)\, \delta^\mu_\nu = \frac{\partial \mathcal{L}(x)}{\partial \vec{\Gamma}_\mu} \vec{\Gamma}_\nu - \mathcal{L}(x)\, \delta^\mu_\nu \tag{30}$$

turns out to be symmetric

$$\Theta^\mu{}_\nu = -\frac{\alpha_f \hbar c}{4\pi} \vec{R}^{\mu\sigma} \vec{R}_{\nu\sigma} - \mathcal{L}(x)\, \delta^\mu_\nu, \tag{31}$$

a property important for the derivation of the angular momentum tensor. This is different to classical electrodynamics where the canonical tensor is gauge dependent and asymmetric.

Global rotations of the soliton field modify the coordinates x_i and the orientations α_i of the local dreibeins. This leads to expressions for the orbital angular momentum

$$L_i = \frac{1}{c} \int \mathrm{d}^3x\, \varepsilon_{ijk} x^j \Theta^{0k}. \tag{32}$$

and the spin

$$\begin{aligned} S_i &= -\frac{1}{c} \int \mathrm{d}^3x \left(\frac{\partial \mathcal{L}}{\partial \partial_0 \alpha_l} \varepsilon_{ilk} \alpha_k \right) = \\ &= -\frac{1}{c} \int \mathrm{d}^3x\, \vec{\pi}^0 \sin\alpha [\cos\alpha\, \vec{n} \times \vec{e}_i + \sin\alpha\, \vec{n} \times (\vec{n} \times \vec{e}_i)]. \end{aligned} \tag{33}$$

Since the analytic solutions of two-soliton systems are not known, the determination of the angular momentum contributions for classical paths can only be determined approximately or numerically. This remains for future work.

5. Summary and outlook

Inspired by the success of the Sine-Gordon model to describe particles by topological solitons of a bosonic field, we tried a generalisation from 1+1D to 3+1D from 1 rotational angle to 3 rotational angles. Since three rotational angles describe rotations in a three-dimensional space, there is the possibility to interpret these rotations as rotations of spatial dreibeins. This leads to the tempting picture that both long-range interactions, gravitation and electromagnetism can be explained by properties of space-time, i.e. using exclusively space and time coordinates. It is an interesting question how far we can drive this idea and where discrepancies to experimental facts get obvious.

Let me enumerate some successes, limitations and open questions.

We describe charges by local dreibeins rotating by 2π along lines crossing the charge centers. Fields of single solitons are connected to the surrounding by lines of constant $\vec{n}$-field. Rotating soliton cores can therefore recover the original configurations only after a 4π-rotations. This indicates the fermionic character of such solitons. By an appropriate Lagrangian solitons are topological stable field configurations which survive scattering processes. Due to the topological character charge is quantised. Such a quantisation is already known from Dirac monopoles. Actually the solitons of this model are dual Dirac monopoles where the two types of singularities of Dirac monopoles were removed. The Dirac string is removed by a transition from the vector potential A_μ to the a three dimensional unit vector field $\vec{n}$ and the singularity in the monopole center is avoided by the transition to an SU(2) field Q. To the author it appears as a nice aspect that in the dual formulation static charges are described by the purely spatial components of the dual field strength tensor (10) whereas "moving" currents lead to the space-time components of the ${}^\star\vec{F}_{\mu\nu}$ and the curvature tensor $\vec{R}_{\mu\nu}$. In classical electrodynamics we find just the opposite correspondence, moving currents are described by static fields and static charges by time components of the fields.

The term quadratic in the field-strength of the Lagrangian (12) seems more natural than the potential term Λ. Nevertheless, such a Λ-term is known from gravitational theory as the cosmological constant. In this model Λ is a cosmological function. One could try to relate the transition from a state with $Q = 1$ and constant $\Lambda = 1/r_0^4$ to $\Lambda = 0$ to inflation known from cosmology and gets a release of energy density of $\alpha_f \hbar c(4\pi r_0^4) = 4.8$ keV/fm^3. The potential term leads to a two-fold degeneracy of the vacuum states. Due to the broken symmetry there are two massless Goldstone bosons described by the $\vec{n}$-component of dreibein rotations. This gives the possibility to interpret photons as Goldstone bosons of rotational symmetry breaking of the vacuum. Lines of constant $\vec{n}$-field can be seen as fibres of a Hopf map from S^3 (R^3 with broken vacuum) to S^2. One could try to follow the conjecture that the Hopf index of this map agrees with the photon number.

The far field of a single soliton, the field $\vec{n}$ covers S^2 once. The two possible topological different coverings determine the sign of the unit charge $\pm e_0$. In this far field limit there is left a rotational invariance around the direction of the $\vec{n}$-field, leading to a U(1) gauge invariance of the dual vector potential C_μ and the dual field strength tensor ${}^\star F_{\mu\nu}$; in this limit both are reduced to one-component fields in internal space as expected for a U(1)-gauge theory. In this model charges and fields are described by the same field $Q(x)$. An artificial separation of solitons and their fields allows to derive the appearance of Coulomb and Lorentz forces. The internal structure of a soliton can be further characterised by the sign of the topological charge $\mathcal{Q} = \pm 1/2$, we called it chirality. The absolute value of $\mathcal{Q}$ can be interpreted as spin quantum number s. The angular momentum property of spin appears in this model only for solitons in orbital motion.

The solitons show all the properties expected for charged particles in a fully relativistic model. The mass of the particles is field energy. Particles experience Lorentz contraction and relativistic mass increase. Moving particles lead to magnetic fields as expected from Lorentz transformations of the electric field. Solitons at rest have a finite size comparable to the classical electron radius. Since they have no real substructure, in central collisions in high energy scattering processes they are expected to be compressed to nearly point-like objects like the Sine-Gordon solitons.

This article investigates on a classical level solitonic excitations of space-time and compares basic properties, like mass, charge and spin of such solitons with particle properties. Quantum fluctuations were not included in these investigations. An appealing method to go beyond the classical behaviour could possibly be indicated by the silicon oil drop experiments of Yves Couder and his group [12]. In these experiments a classical particle interacts with a wave-field which results from a superposition of an external field and the field created by the particle. Under appropriate conditions this resulting pilot-wave field leads to interference [13], tunnelling [14] and spontaneous quantisation of closed orbits [15]. Transferring this experience to solitons would mean that particles on their classical path are disturbed by, and interacting with, a subquantum medium. Such a medium must not transfer energy and momentum by friction. Deceleration by friction processes can only be avoided by a medium propagating with the speed of light which does not prefer special reference systems. Further, to get interference phenomena with classical solitons resonance processes between the subquantum medium and solitons could be essential. Since solitons are extended objects one could imagine that they select and interact with certain frequencies from a wave-field.

Due to the low number of degrees of freedom electromagnetic fields are more restricted than in classical electrodynamics. On the other hand besides solutions of the Maxwell equations there is a possibility for non-vanishing non-quantised magnetic currents [10, 11]. Further one can expect α-waves propagating through the vacuum [16]. These presently unknown excitations one can try to shift to the subquantum medium.

Finally one can ask about strong and weak interactions. It is obvious that to include these interactions in the model an extension by further field degrees of freedom is necessary. If one adheres to the idea that the fields are properties of space-time this would lead to further dimensions which would be hidden inside of strongly interacting particles.

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