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2014 J. Phys.: Conf. Ser. 504 012005

(http://iopscience.iop.org/1742-6596/504/1/012005)

Nonlinear Riccati equations as a unifying link between linear quantum mechanics and other fields of physics

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Abstract. Theoretical physics seems to be in a kind of schizophrenic state. Many phenomena in the observable macroscopic world obey nonlinear evolution equations, whereas the microscopic world is governed by quantum mechanics, a fundamental theory that is supposedly linear. In order to combine these two worlds in a common formalism, at least one of them must sacrifice one of its dogmas. I claim that linearity in quantum mechanics is not as essential as it apparently seems since quantum mechanics can be reformulated in terms of nonlinear Riccati equations. In a first step, it will be shown where complex Riccati equations appear in time-dependent quantum mechanics and how they can be treated and compared with similar space-dependent Riccati equations in supersymmetric quantum mechanics. Furthermore, the time-independent Schrödinger equation can also be rewritten as a complex Riccati equation. Finally, it will be shown that (real and complex) Riccati equations also appear in many other fields of physics, like statistical thermodynamics and cosmology.

1. Introduction

In the beginning of the 20th century, one of the biggest mysteries in physics was the observation of material systems behaving in certain experiments (as expected) like particles, in others, however, (unexpectedly) like waves. This puzzle of wave-particle duality was finally solved by quantum mechanics, developed around the same time by Heisenberg and Schrödinger, whereby Schrödinger's wave mechanical formulation [1] was more popular from the beginning because it used partial differential equations, which were more familiar to most physicists than Heisenberg's matrix mechanics, though both are essentially equivalent. As Schrödinger's original formulation is based on classical Hamilton–Jacobi mechanics it bears similar features to its classical counterpart, like conservation of energy (the operator corresponding to the Hamiltonian function is Hermitian and a constant of motion) and time-reversal symmetry of the dynamics (the time evolution is described by unitary transformations). Since Schrödinger's equation is linear and has features of a wave equation (for a complex quantity!) the superposition principle applies. This is advantageous for computational purposes (besides being suitable for the above-mentioned wave properties of material systems). As almost the last 90 years have shown, quantum mechanics is not only scientifically, but also economically, the most successful and influential theory so far.

Towards the end of the 20th century, another development in physics became quite popular, namely nonlinear (NL) dynamics [2] (also because of the aesthetically-appealing pictures of so-called fractals [3] that were actually merely a side-product). This theory is able to describe many of the phenomena observed in our every day surroundings like the weather, growth processes, etc. It also includes such basic experiences like evolution with a direction of time and dissipation of energy; both aspects are, as mentioned above, absent in quantum (and classical) mechanics in the original canonical form. In addition, the laws of NL dynamics are usually scale invariant; the size of the system does not matter only relative changes are relevant. There are phenomena like self-similarity, fractals, Mandelbrot sets, etc. [4].

Apart from self-similarity, i.e., finding the same structures at different scales, a striking property of fractals is the appearance of spiral forms, i.e., structures whose radius changes while the angle rotates. This combination of radial and angular motion is well known in nature as anyone can verify by looking at the shell of a nautilus or the horn of a ram. In these cases one actually has a "frozen" picture of the evolution and the radius is in fact a measure of time (in the case of the nautilus, the larger the distance from the centre, the older the part of the shell is; with the ram, the opposite applies). The shape of these spirals usually takes the form of a logarithmic spiral, where the radius grows (or shrinks) exponentially.

So, at the beginning of the 21st century, we are faced with a rather schizophrenic situation:

- Quantum mechanics (and other so-called fundamental theories) are

- reversible
- conservative
- linear.

- Observable (macroscopic) nature (as described, e.g., by NL dynamics) is

- irreversible
- dissipative
- nonlinear.

- In addition, there are fields of physics that are not naturally compatible with quantum mechanics, like

- thermodynamics
- cosmology.

Now the question arises: is it possible to find a formulation that is able to unify all these different aspects of physics? At first sight, the answer seems to be negative since linearity of quantum mechanics, by definition, contradicts NL dynamics. But, could it be that quantum mechanics is only a NL theory in disguise (in other words a linearized form of an underlying, more fundamental NL theory)? It might even be only one example of many that can be traced back to the same NL theory. So the question is not so much one of quantum mechanics emerging from a more fundamental "sub-quantum" theory, but rather one of being a (linearized) version of a formal NL theory where this theory applies to many other (in the luckiest case all) fields of physics but appearing in different forms concealing the similarity of the underlying (NL) structure.

In the following, it shall be shown how a (complex) Riccati equation might be a candidate for such a fundamental NL theory. Not only does this equation possess a kind of superposition principle, due to its linearizability, but it also has properties usually common only in NL systems, like sensitivity to initial conditions, bifurcation, and so on.

In Section 2, it will be shown that complex Riccati equations already occur in conventional time-dependent (TD) quantum mechanics and formal similarities (pertaining to the sensitivity to initial conditions) with supersymmetric quantum mechanics will be emphasized. Section 3 presents a short overview of similar formal structures that also exist in time-independent (TI) quantum mechanics. Dissipative versions of both aspects exist, but since they have been discussed in previous articles, details will not be given here but can be found in [5]. Section 4 comprises a comparison of the formal structures found in Section 2 with similar ones in statistical thermodynamics and cosmology.

2. Complex Riccati equations related to the time-dependent Schrödinger equation

In the following, one-dimensional problems with exact analytic solutions of the TD Schrödinger equation (SE) in the form of Gaussian wave packets will be considered, particularly the free motion (potential $V(x) = 0$) and the harmonic oscillator (HO) ($V = \frac{m}{2}\omega^2 x^2$) with constant frequency, $\omega = \omega_0$, or TD frequency, $\omega = \omega(t)$. In these cases, the solution of the TDSE (here for the HO, the case $V = 0$, in the following, is always obtained in the limit $\omega \to 0$)

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) \;=\; \left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega^2 x^2 \right\} \Psi(x,t) \tag{1}$$

(where $\hbar = \frac{h}{2\pi}$ with $h =$ Planck's constant) can be written as

$$\Psi(x,t) \;=\; N(t) \exp\left\{ i \left[y(t)\, \tilde{x}^2 + \frac{<p>}{\hbar}\, \tilde{x} + K(t) \right] \right\} \tag{2}$$

with the shifted coordinate $\tilde{x} = x - <x> = x - \eta(t)$, where the mean value $<x> = \int_{-\infty}^{+\infty} \Psi^* x \Psi dx = \eta(t)$ corresponds to the classical trajectory, $<p> = m\dot{\eta}$ represents the classical momentum and the coefficient of the quadratic term in the exponent, $y(t) = y_R(t) + iy_I(t)$, is a complex function of time. The (possibly TD) normalization factor $N(t)$ and the purely TD function $K(t)$ in the exponent are not relevant to the following discussion.

The equations of motion for $\eta(t)$ and $y(t)$, or $\left(\frac{2\hbar}{m} y = \mathcal{C}\right)$ that are obtained by inserting WP (2) into Eq. (1) are important for our purpose and are given by

$$\ddot{\eta} + \omega^2 \eta = 0 \;, \tag{3}$$

and

$$\dot{\mathcal{C}} \;+\; \mathcal{C}^2 \;+\; \omega^2 \;=\; 0 \;, \tag{4}$$

where overdots denote derivatives with respect to time. The Newtonian equation (3) simply means that the maximum of the WP, located at $x = <x> = \eta(t)$, follows the classical trajectory. The equation for the quantity $\frac{2\hbar}{m} y(t) = \mathcal{C}$ has the form a of a *complex* NL Riccati equation and describes the time-dependence of the WP width that is related to the position uncertainty via $y_I = \frac{1}{4<\tilde{x}^2>}$ with $<\tilde{x}^2> = <x^2> - <x>^2$ being the mean square deviation of position.

The dynamics of this quantity can be described more conveniently by introducing a new (real) variable $\alpha(t)$ via $\mathcal{C}_I = \left(\frac{2\hbar}{m} y_I\right) = \frac{1}{\alpha^2}$. Inserting this into the imaginary part of Eq. (4) allows one to determine the real part of the variable as $\mathcal{C}_R = \left(\frac{2\hbar}{m} y_R\right) = \frac{\dot{\alpha}}{\alpha}$, which, when inserted into the real part of (4) together with the above definition of $\mathcal{C}_I$, finally turns the complex Riccati equation into the real NL so-called Ermakov equation [1] for $\alpha(t)$,

$$\ddot{\alpha} + \omega^2 \alpha = \frac{1}{\alpha^3} \;. \tag{5}$$

[1] This equation was studied already in 1874 by Adolph Steen [6]. However, Steen's work was ignored by mathematicians and physicists for more than a century because it was published in Danish in a journal usually not containing many articles on mathematics. An English translation of the original paper [7] is available and generalizations can be found in [8].

It had been shown by Ermakov [9] in 1880, i.e., 45 years before quantum mechanics was formulated by Schrödinger and Heisenberg, that from the pair of equations (3) and (5), coupled via ω^2, by eliminating ω^2 from the equations, a dynamical invariant, the Ermakov-invariant

$$I_L = \frac{1}{2}\left[(\dot{\eta}\alpha - \eta\dot{\alpha})^2 + \left(\frac{\eta}{\alpha}\right)^2\right] = \text{const.} \tag{6}$$

can be obtained (this invariant was rediscovered by several authors, also in a quantum mechanical context; see, e.g., [10, 11, 12]).

In the following, the remarkable properties of this invariant shall not be further considered (for details, see [13]), but a different way of treating the (inhomogeneous) Riccati equation shall be discussed. Instead of transforming it into the (real) NL Ermakov equation (5), it can be solved directly by transforming it into a homogeneous NL (complex) Bernoulli equation if a particular solution $\tilde{\mathcal{C}}$ of the Riccati equation is known. The general solution of Eq. (4) is then given by $\mathcal{C} = \tilde{\mathcal{C}} + \mathcal{V}(t)$ where $\mathcal{V}(t)$ fulfils the Bernoulli equation

$$\dot{\mathcal{V}} + 2\,\tilde{\mathcal{C}}\mathcal{V} + \mathcal{V}^2 = 0\ . \tag{7}$$

The coefficient of the linear term depends on the particular solution $\tilde{\mathcal{C}}$. Equation (7) can be linearized via $\mathcal{V} = \frac{1}{\kappa(t)}$ to yield

$$\dot{\kappa} - 2\,\tilde{\mathcal{C}}\,\kappa = 1\ , \tag{8}$$

which can be solved straightforwardly. For constant $\tilde{\mathcal{C}}$, $\kappa(t)$ can be expressed in terms of exponential or hyperbolic functions (for real $\tilde{\mathcal{C}}$) or trigonometric functions (for imaginary $\tilde{\mathcal{C}}$). In this case, $\mathcal{C}$ can be written as

$$\mathcal{C}(t) = \tilde{\mathcal{C}} + \frac{e^{-2\,\tilde{\mathcal{C}}t}}{\frac{1}{2\tilde{\mathcal{C}}}\left(1 - e^{-2\,\tilde{\mathcal{C}}t}\right) + \kappa_0}\ . \tag{9}$$

For $\tilde{\mathcal{C}}$ being TD, $\kappa(t)$ and hence $\mathcal{V}$ can be expressed in terms of $\mathcal{I}(t) = \int^t dt'\ \mathrm{e}^{-\int^{t'} dt" 2\tilde{\mathcal{C}}(t")}$. So the general solution of Eq. (4) can be written as

$$\mathcal{C}(t) = \tilde{\mathcal{C}} + \frac{d}{dt}\ln\left[\kappa_0 + \mathcal{I}(t)\right]\ , \tag{10}$$

defining a one-parameter family of solutions depending on the (complex) initial value of $\kappa_0 = \mathcal{V}_0^{-1}$ as parameter. To realize the strong qualitative influence of the initial value κ_0 on the solution of the Riccati equation (which is not surprising since this is a NL differential equation), a comparison with supersymmetric (SUSY) quantum mechanics [14, 15, 16, 17] might be quite useful.

In SUSY quantum mechanics, the Hamiltonian can be represented by a 2 $\times$ 2 diagonal matrix where the potentials V_i of the Hamiltonian operators $H_i = -\frac{\hbar^2}{2m}\frac{d^2}{dx_2} + V_i$ (i= 1,2) on the diagonal are determined by a (real) Riccati equation for the so-called "superpotential" $W(x)$,

$$V_{1/2} = \frac{1}{2}\left[W^2 \mp \frac{\hbar}{\sqrt{m}}\frac{d}{dx}W\right]\ , \tag{11}$$

and differ only by the sign of the term depending on the derivative of $W(x)$. Again, one can try to solve this Riccati equation by reducing it to a Bernoulli equation using a particular solution

$\tilde{W}$. In the case of the HO, this particular solution is given by $\tilde{W} = \sqrt{m}\omega_0\, x$, leading to the two potentials

$$\tilde{V}_{1/2} \;=\; \frac{m}{2}\,\omega_0^2\,x^2 \;\mp\; \frac{\hbar}{2}\,\omega_0\;, \tag{12}$$

i.e., essentially the parabolic harmonic potential, only shifted by minus/plus the ground state energy $\frac{\hbar}{2}\omega_0$. The general solution can again be written in the form $W(x) = \tilde{W}(x) + \Phi(x)$ where $\Phi(x)$ must now fulfil the Bernoulli equation

$$\frac{\hbar}{\sqrt{m}}\,\frac{d}{dx}\Phi \;+\; 2\,\tilde{W}\,\Phi \;+\; \Phi^2 \;=\; 0 \tag{13}$$

(written here for the plus sign of the derivative). This can be solved in the same way by linearization to finally yield the general solution

$$W(x) \;=\; \tilde{W}(x) \;+\; \frac{\hbar}{\sqrt{m}}\,\frac{d}{dx}\,\ln\left[\varepsilon + \mathcal{I}(x)\right]\,. \tag{14}$$

The integral $\mathcal{I}(x)$ is formally identical to the one in the TD case, only t must be replaced by x and $\tilde{C}$ by $\tilde{W}$. Also this solution depends on a (this time real) parameter ε (corresponding to $\Phi^{-1}(0)$). This generalized $W(x)$ gives rise to a one-parameter family of isospectral potentials (e.g., for i = 1)

$$V_1(x;\varepsilon) \;=\; \tilde{V}_1(x) \;-\; \frac{\hbar^2}{m}\,\frac{d^2}{dx^2}\,\ln\left[\varepsilon + \mathcal{I}(x)\right]\,, \tag{15}$$

i.e., potentials with different shapes, but the same energy spectrum, namely that of the HO, only with ground state energy equal to zero (apart from $\varepsilon = 0$, where this state is missing).

The shape of the potentials is now, however, unlike the parabolic harmonic potential, no longer symmetric under the exchange $x \to -x$; there is even a second minimum showing up for negative x whose depth increases with decreasing ε (for details, see [16]). Only for $\varepsilon \to \infty$, the ln-term vanishes and the parabolic potential $\tilde{V}_1(x)$ is re-gained. So, obviously the parameter ε can have drastic qualitative consequences for the solution of the Riccati equation. The same can also happen in the above-mentioned TD case when κ_0 is varied.

Finally, a third way to treat the complex Riccati equation (4) shall be mentioned. Using the ansatz

$$\mathcal{C} \;=\; \left(\frac{2\hbar}{m}y\right) \;=\; \frac{\dot{\lambda}}{\lambda}\;, \tag{16}$$

with complex $\lambda(t)$ linearizes Eq. (4) to yield the complex Newtonian equation

$$\ddot{\lambda} + \omega^2(t)\lambda = 0\;. \tag{17}$$

Writing λ in terms of real and imaginary parts, i.e., $\lambda = u + i\,z$ would just result in two identical Newtonian equations for u and z where it can be shown [18] that, up to a constant factor, z is identical to the classical trajectory $\eta(t)$. More information can be gained by writing λ in polar form as $\lambda = \alpha e^{i\varphi}$. From definition (16) one then obtains

$$\mathcal{C} \;=\; \frac{\dot{\alpha}}{\alpha} + i\,\dot{\varphi}\;. \tag{18}$$

Inserting $\mathcal{C}$ in this form into the complex Riccati equation turns the imaginary part of this equation into

$$\dot{\varphi} = \frac{1}{\alpha^2}\;, \tag{19}$$

which looks like the conservation of angular momentum for the motion of $\lambda(t)$ in the complex plane (and also agrees with the definition of $\mathcal{C}_I$ at the beginning of this section). The real part of the Riccati equation then turns simply into the Ermakov equation (5) for $\alpha(t)$.

3. Complex Riccati equations related to the time-independent Schrödinger equation

We have seen in the TD case that the real and imaginary parts, or phase φ and amplitude α, of the complex variable $\lambda(t) = \alpha\ e^{i\varphi}$ which fulfils the linear equation (17), obtained via Eq. (16) from the Riccati equation (4), are not independent of each other but coupled via the conservation law (19). A similar situation exists when considering the TISE, but now in the space-dependent case.

This can be shown using Madelung's hydrodynamic formulation of quantum mechanics [19] where the wave function is written in polar form as

$$\Psi(\mathbf{r},t) = \varrho^{1/2}(\mathbf{r},t)\ \exp\left(\frac{i}{\hbar}S(\mathbf{r},t)\right) \tag{20}$$

with the square root of the probability density $\varrho = \Psi^*\Psi$ as amplitude and $\frac{1}{\hbar}S$ as phase ($\mathbf{r}$ is the position vector in three dimensions).

Inserting this form into the TDSE (1) (now in three dimensions), and replacing $\frac{\partial}{\partial x}$ by the nabla operator ∇), leads to a modified Hamilton–Jacobi equation for the phase,

$$\frac{\partial}{\partial t}S + \frac{1}{2m}(\nabla S)^2\ + V\ - \frac{\hbar^2}{2m}\frac{\Delta\varrho^{1/2}}{\varrho^{1/2}}\ =\ 0\ , \tag{21}$$

and a continuity equation for the amplitude,

$$\frac{\partial}{\partial t}\varrho\ +\ \frac{1}{m}\nabla(\varrho\ \nabla S) = 0\ . \tag{22}$$

Already here, the coupling of phase and amplitude can be seen clearly since the Hamilton–Jacobi equation for the phase S contains a term (misleadingly called "quantum potential", V_{qu}) depending on ϱ, and the continuity equation for the density ϱ contains ∇S. It can be shown that also in the TI case this coupling is not arbitrary but related to a conservation law.

In 1994, G. Reinisch [20] did this in a NL formulation of TI quantum mechanics. Since in this case $\frac{\partial}{\partial t}\varrho = 0$ and $\frac{\partial}{\partial t}S = -E$ are valid, the continuity equation (22) (we now use the notation $\varrho^{1/2} = |\Psi| = a$) turns into

$$\nabla(a^2\nabla S)\ =\ 0 \tag{23}$$

and the modified Hamilton–Jacobi equation into

$$-\ \frac{\hbar^2}{2m}\Delta a\ +\ (V-E)\ a\ =\ -\ \frac{1}{2m}(\nabla S)^2 a\ . \tag{24}$$

Equation (23) is definitely fulfilled for $\nabla S = 0$, turning (24) into the usual TISE for the real wave function $a = |\Psi|$ with position-independent phase S. (N.B.: the kinetic energy term divided by a is just identical to V_{qu}!)

However, Eq. (23) can also be fulfilled for $\nabla S \neq 0$ if only the conservation law

$$\nabla S\ =\ \frac{C}{a^2} \tag{25}$$

is fulfilled with constant (or, at least, position-independent) C.

This relation now shows explicitly the coupling between phase and amplitude of the wave function and is equivalent to Eq. (19) in the TD case. Inserting (25) into the rhs of Eq. (24) changes this into the Ermakov equation

$$\Delta a\ +\ \frac{2m}{\hbar^2}(E-V)\ a\ =\ (\frac{1}{\hbar}\nabla S)^2\ a\ =\ \left(\frac{C}{\hbar}\right)^2\frac{1}{a^3}\ , \tag{26}$$

equivalent to Eq. (5) in the TD case.

Returning to the method described in [20], so far the energy E occurring in Eq. (26) is still a free parameter that can take any value. However, solving this equation numerically for arbitrary values of E leads, in general, to solutions a that diverge for increasing$x$. Only if the energy E is appropriately tuned to any eigenvalue E_n of the TISE (see Eq. (28), below) does this divergence disappear and normalizable solutions can be found. So, the quantization condition that is usually obtained from the requirement of the truncation of an infinite series in order to avoid divergence of the wave function is, in this case, obtained from the requirement of nondiverging solutions of the NL Ermakov equation (26) by variation of the parameter E. This has been numerically verified in the case of the one-dimensional HO and the Coulomb problem and there is the conjecture that this property is "universal" in the sense that it does not depend on the potential V (see [20, 21]).

The corresponding complex Riccati equation is now given by

$$\nabla\left(\frac{\nabla\Psi}{\Psi}\right) + \left(\frac{\nabla\Psi}{\Psi}\right)^2 + \frac{2m}{\hbar^2}(E-V) = 0 \tag{27}$$

with the complex variable $\mathcal{C} = \left(\frac{\nabla\Psi}{\Psi}\right) = \frac{\nabla a}{a} + i\frac{1}{\hbar}\nabla S$ which corresponds to $\left(\frac{2\hbar}{m}y\right) = \frac{\dot\lambda}{\lambda} = \frac{\dot\alpha}{\alpha} + i\,\dot\varphi$ in the TD problem.

It is possible to show straightforwardly that Eq. (27) can be linearized to yield the usual TISE

$$-\frac{\hbar^2}{2m}\Delta\Psi + V\,\Psi = E\,\Psi\,, \tag{28}$$

but in this form, the information on the coupling of phase and amplitude, expressed by Eq. (25) and originating from the quadratic NL term in Eq. (27), gets lost.

4. Riccati equations in other fields of physics

Let us return to solution (9) of the Riccati equation (4), but now for real $\mathcal{C}$, and rewrite it in the form

$$\mathcal{C}(t) = \tilde{\mathcal{C}} + \frac{2\tilde{\mathcal{C}}\,e^{-2\tilde{\mathcal{C}}t}}{\kappa_0\,2\tilde{\mathcal{C}} + \left(1-e^{-2\tilde{\mathcal{C}}\,t}\right)} = \tilde{\mathcal{C}} + \frac{2\tilde{\mathcal{C}}}{\kappa_0\,2\tilde{\mathcal{C}}\;e^{2\tilde{\mathcal{C}}t} + \left(e^{2\tilde{\mathcal{C}}\,t} - 1\right)}\,. \tag{29}$$

For the choice $\kappa_0 = 0$, this turns into

$$\mathcal{C}(t) = \tilde{\mathcal{C}} + \frac{2\tilde{\mathcal{C}}}{\left(e^{2\tilde{\mathcal{C}}\,t} - 1\right)} = \tilde{\mathcal{C}}\,\coth\,\tilde{\mathcal{C}}t\,. \tag{30}$$

Replacing time t by $t \to \frac{1}{kT} = \beta$ (with k = Boltzmann's constant) and the constant particular solution by $\tilde{\mathcal{C}} = \frac{\hbar}{2}\,\omega$, one obtains

$$\frac{\hbar}{2}\,\omega + \frac{\hbar\,\omega}{e^{\hbar\omega/kT} - 1} = \frac{\hbar}{2}\,\omega\,\coth\left(\frac{\hbar\omega}{2kT}\right) = \,<E>_{th} \tag{31}$$

which is the expression known from statistical thermodynamics for the average energy of a single oscillator in thermal equilibrum [22]. The first term on the lhs is just the ground state energy of the harmonic oscillator, the second is equal to Planck's distribution function for the black body radiation. This type of relation between Eq. (31) and the Riccati equation has also been found by Rosu et al [23]. Equation (31) can also be expressed in terms of the partition function $Z \;=\; \sum_n \; e^{-n\hbar\omega\beta} \;=\; \frac{1}{1\,-\,e^{\hbar\omega\beta}}$ as

$$< E >_{th} \;=\; \frac{\hbar}{2}\,\omega \;+\; \frac{\frac{\partial}{\partial\beta}\;Z^{-1}}{Z^{-1}} \;=\; \frac{\hbar}{2}\,\omega \;-\; \frac{\partial}{\partial\beta}\;\ln\;Z\;. \tag{32}$$

The Riccati equation corresponding to solution (30) in the form (31) can be written as

$$\mathcal{C}' \;+\; \mathcal{C}^2 \;-\; \tilde{\mathcal{C}}^2 \;=\; 0 \tag{33}$$

with $\mathcal{C} \;=\; \mathcal{C}(\beta)$ depending on the variable $\beta \;=\; \frac{1}{kT}$, prime denoting derivative with respect to this variable and $\tilde{\mathcal{C}}$, as mentioned above, being the ground state energy, $\tilde{\mathcal{C}} \;=\; \frac{\hbar}{2}\omega$. The inverse quantity of $\mathcal{C}$, multiplied by $-\,\tilde{\mathcal{C}}^2$, i.e., $\mathcal{K} \;=\; -\;\tilde{\mathcal{C}}^2\;\mathcal{C}^{-1}$ fulfils

$$-\,\mathcal{K}' \;+\; \mathcal{K}^2 \;-\; \tilde{\mathcal{C}}^2 \;=\; 0\;, \tag{34}$$

i.e., also a Riccati equation but now with negative sign for the derivative term and the solution

$$\mathcal{K}\left(\frac{1}{kT}\right) \;=\; \frac{\hbar}{2}\,\omega \;-\; \frac{\hbar\;\omega}{e^{\hbar\omega/kT}\;+\;1} \;=\; \frac{\hbar}{2}\,\omega\;\tanh\left(\frac{\hbar\omega}{2kT}\right)\;. \tag{35}$$

¿From the ground state energy in this case a term is subtracted that represents a Fermi–Dirac distribution, whereas in the solution for $\mathcal{C}$, a term was added to the ground state energy that represents a Bose–Einstein distribution. So both quantum statistics can be obtained from the solution of the Bernoulli equations derived from the Riccati equations for $\mathcal{C}\left(\frac{1}{kT}\right)$ and (essentially) its inverse quantity.

Another connection between Bose–Einstein systems and a Riccati equation (in this case a complex one) can be found if a Bose–Einstein condensate (BEC) is described in the mean field approximation by a macroscopic wave packet for the condensate, Ψ, which obeys the Gross–Pitaevskii equation

$$i\hbar\frac{\partial}{\partial t}\Psi = \left\{-\frac{\hbar^2}{2m}\Delta + V(\mathbf{r},t) + g|\Psi|^2\right\}\Psi \tag{36}$$

where $V(\mathbf{r},t)$ represents the trapping potential and shall be given by $V(\mathbf{r},t) \;=\; \frac{m}{2}\omega^2(t)r^2$ with $r = |\mathbf{r}| =$ absolute value of the vector $\mathbf{r}$ and TD frequency $\omega = \omega(t)$. Although this equation cannot be solved analytically, the dynamics of the BEC described by it can be obtained from a set of coupled differential equations for so-called moments that essentially represent the width, radial momentum and energy of the wave packet (for details, see [24]). The important point is that these three moments fulfil a closed set of three TD differential equations that (with the help of a conserved quantity K that corresponds to the conserved angular momentum in the complex plane, discussed in section 2) can be reduced to one equation for the square root of the width (denoted by $X(t)$ and corresponding to $\alpha(t)$ in section 2) with the form of an Ermakov equation,

$$\ddot{X} + \omega^2(t)X = \frac{k}{X^3}\;. \tag{37}$$

Since Eq. (37) is, as we have seen in section 2, only a different way of writing a complex Riccati equation by making use of a conservation law, also this BEC can be described by a, now complex, Riccati equation.

Finally, another example shall be given where a physical system can be described by a (complex) Riccati equation, or its equivalent, a real Ermakov equation, now not for a microscopic but for a really macroscopic system. For this purpose, we switch to cosmology and apply to Einstein‘s field equations the cosmological principle, i.e., the assumption of a spatially-homogeneous and isotropic universe. In this case, the Robertson–Walker metric applies [25] that contains the scale factor $a(t)$ (so to say, the radius of the universe) and the curvature k that can attain the values 0 for a flat or $\pm$ 1 for a closed or open universe. This finnaly leads to the Friedman–Lemaitre equations

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{2}{3}\,\varrho - \frac{k}{a^2} \tag{38}$$

and

$$\dot{\varrho} = -3\,H\,(\varrho + p) \tag{39}$$

with $H = \frac{\dot{a}}{a}$ = Hubble parameter; overdots denote derivatives with respect to cosmic proper time, ϱ = energy density and p = pressure (where $4\pi G = c = 1$ and the cosmological constant $\Lambda = 0$ have been set).

Assuming the matter source as a self-interacting scalar field $\Phi = \Phi(t)$, the energy density ϱ and pressure p can be written as

$$\varrho = \frac{1}{2}\left(\frac{d}{dt}\Phi\right)^2 + U(t)\;, \tag{40}$$

$$p = \frac{1}{2}\left(\frac{d}{dt}\Phi\right)^2 - U(t)\;. \tag{41}$$

Taking the time-derivative of (38) and inserting (39) using (40) and (41) leads to

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = -\left(\frac{d}{dt}\Phi\right)^2 + \frac{k}{a^2}\;. \tag{42}$$

Introducing a new time variable (conformal time τ) via $\frac{d}{dt} = a\,\frac{d}{d\tau}$, Eq. (42) can be rewritten as an Ermakov equation,

$$\frac{d^2}{d\tau^2}\,a + \left(\frac{d}{d\tau}\Phi\right)^2 a = \frac{k}{a^3}\;, \tag{43}$$

which is equivalent to a Riccati equation for the complex quantity $\mathcal{C} = \left(\frac{\frac{d}{d\tau}a}{a} + i\,\frac{1}{a^2}\right)$. The above derivation of the Ermakov equation in comparison with the afore-mentioned BEC had been given by Lidsey in [26]. Everything said about the complex or real Ermakov equation in section 2 can also be applied to this system, e.g., corresponding creation and annihilation operators and coherent states can be defined (see [13]).

5. Conclusions and perspectives

It has been shown in section 2 that the information about the dynamics of a quantum mechanical wave packet, i.e. the equations of motion for the time-evolution of its maximum and width can not only be obtained from the solution of the TDSE, but equally well from a complex NL Riccati equation. The square root of the inverse of the imaginary part of the quantity fulfilling this equation, $\alpha(t)$, is (up to a constant factor) just the width of the wave packet, fulfilling a so-called Ermakov equation. The classical particle aspect that is reflected by the Newtonian equation determining the motion of the wave packet maximum can also be obtained, using that the complex Riccati equation can be linearized by the ansatz $\mathcal{C} = \frac{\dot{\lambda}}{\lambda}$ to a Newtonian equation for the complex quantity $\lambda(t)$, which can be written as $\lambda = u + z = \alpha\,e^{i\varphi}$. From the knowledge of α (as solution of the Ermakov equation) and with $\dot{\varphi} = \frac{1}{\alpha^2}$, $\varphi(t)$ can be determined and hence

$\lambda(t)$ is obtained. Knowing that $z \propto \eta(t)$, the imaginary part of λ provides (up to a constant factor) immediately the classical trajectory $\eta(t)$.

Formal similarities exist between this TD problem and SUSY quantum mechanics, ehere (real) position dependent Riccati equations play a similar role and the construction of isospectral potentials show the importance of parameters like initial conditions.

In Section 3 it has been shown that the Riccati formalism established for the TDSE can also be applied to the TISE if the TD complex quantity $\mathcal{C} = \frac{\dot{\lambda}}{\lambda}$ is replaced by the space-dependent complex quantity $\frac{\nabla\Psi}{\Psi}$ for a complex wave function Ψ. In a certain way, this looks like a complex version of SUSY where not only the logarithmic derivative of the ground state is considered as a variable for a Riccati equation, but also any (even complex) excited state may fulfil a now complex Riccati equation.

The flexibility of the Riccati formalism is illustrated in section 4 where we see that it is not only restricted to quantum mechanics. Replacing time t with (actually imaginary) "time" $\frac{\hbar}{kT}$ leads to well-known expressions from statistical thermodynamics as solutions of Riccati equations that are also able to distinguish between bosonic and fermionic properties. While this example was still dealing with real Riccati equations, in the next example it was shown that also the dynamics of a BEC can be described by a complex Riccati or the corresponding Ermakov equation. Finally, we went from the microscopic scale to the really macroscopic one, to cosmology and demonstrated that the Friedman–Lemaitre equations, under certain assumptions, describe the dynamics of our universe and can be written in terms of a real Ermakov or equivalent complex Riccati equation.

There are also many more examples, from electrodynamics via quantum optics to soliton theory, etc., where the Riccati equation allows for a unifying formulation with the fields mentioned above as well as others. In conclusion therefore, one can say that the NL version of quantum mechanics based on a (complex) Riccati equation is able to cover all phenomena of standard quantum mechanics (including the superposition principle, due to its linearisablity). Due to the sensitivity of NL differential equations to the initial conditions, it can further provide additional information that gets lost in the linearised form. The complex form of the Riccati equation also supplies (via its imaginary part) a new conservation law that resembles the conservation of angular momentum, but now for the motion in the complex plane; something closely related to the quantum property spin.

It should also be mentioned again that effects like dissipation and irreversibility can easily be included into the Riccai formalism. In the TD case, only an additional term linear in $\mathcal{C}$ appears in Eq. (4). Since in the solution via the homogeneous Bernoulli equation (7) a linear term already occurs, only the coefficient of this term changes. However, in the TD quantum mechanical case, this leads to bifurcation, i.e., two qualitatively different solutions for the dynamics of the wave packet width, a phenomena well-known in NL dynamics but alien to quantum mechanics (for details see [27]). For the damped HO, $\lambda(t)$ is no longer moving on a circle in the complex plane but on a spiral with the radius decreasing exponentially.

So maybe it is not so much the question of quantum mechanics emerging from an underlying theory that possesses properties like non-equilibrium thermodynamics, hybrid-mechanics or other structures, but that of quantum mechanics, thermodynamics, nonlinear dynamics and other fields of physics somehow being traced back to a formalism based on complex NL Riccati equations or generalizations thereof.

Acknowledgments

The author would like to thank the organizers and sponsors, particularly Gerhard Grössing, for the generous invitation to participate in this very interesting and stimulating conference.

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